The generic symplectic C^1-diffeomorphisms of 4-dimensional symplectic manifolds are hyperbolic, partially hyperbolic or have a completely elliptic periodic point

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Abstract

We prove that if (M, ω) is a connected and compact 4-dimensional symplectic manifold, there exist three open sets U_1, U_2, U_3 of $\text{Diff}^1_\omega(M)$ (for the C^1-topology) such that :

- $U_1 \cup U_2 \cup U_3$ is dense in $\text{Diff}^1_\omega(M)$;
- $f \in U_1$ if and only if f is Anosov and transitive;
- $f \in U_2$ if and only if f is partially hyperbolic;
- $f \in U_3$ if and only if f has a stable completely elliptic periodic point.

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Contents

1 Introduction.

Let (M, ω) be a compact riemannian symplectic manifold and let $\text{Diff}^1_\omega(M)$ be the set of symplectic C^1-diffeomorphisms of M, endowed with the C^1 topology.

The sets of diffeomorphisms which will interest our are the sets which are large in the sense of the Baire category (for the C^1 topology). One well known such set is the set of symplectic Anosov diffeomorphisms. This set is open in the C^1 topology, but it is not dense in $\text{Diff}^1_\omega(M)$. In fact, S. Newhouse proved in [7] the following result :

Theorem 1 (S. Newhouse) *Let (M, ω) a symplectic compact and connected manifold and let $f \in \text{Diff}^1_\omega(M)$ be a symplectic diffeomorphism which is not Anosov. Then for any U open non empty subset of M and any neighbourhood $\mathcal{V}$ of f (in C^1 topology), there exists $g \in \mathcal{V}$ having a 1-elliptic periodic point in U.*

In this article, we shall give a refinement of this last result in the case : $\dim M = 4$. To do that, we shall replace the notion of Anosov diffeomorphism by a weaker one : the notion of partially hyperbolic diffeomorphism, and replace the notion of 1-elliptic periodic points by the notion of completely elliptic periodic points. Let us give the definition contained in [2] :

DEFINITION : Let $f \in \text{Diff}^1_\omega(M)$ (we assume that M is compact). Let K be a non-empty subset of M invariant by f, and $TM_{|K} = E \oplus F$ a splitting of $TM_{|K}$ invariant by Df, whose fibers have constant dimension.
Then, $E \oplus F$ is a (ℓ-)dominated splitting (of f on K) if there exists $\ell \in \mathbb{N}^*$ such that : $\forall x \in K, \|Df^\ell(x)_{|E}\|.\|Df^{-\ell}(f^\ell(x))_{|F}\| < \frac{1}{2}$.
This is written : $E \prec F$ or $E \prec_\ell F$.

This definition is quite different from the one of hyperbolicity because it doesn't require that $Df_{|E}$ is contracting or $Df_{|F}$ is expanding.
Let us recall some properties of the dominated splittings which are proved in [2] :

1 every dominated splitting on K can be uniquely extended to the closure $\bar{K}$ of K;

2 every dominated splitting on K is continuous;

3 if there exists a ℓ-dominated splitting of f on K, there exists an open neighbourhood U of $\bar{K}$ in M and an open neighbourhood $\mathcal{V}$ of f in C^1 topology such that for every g in $\mathcal{V}$, the maximal invariant set (under g) contained in U has a ℓ-dominated splitting (and this dominated splitting has constant dimension).

REMARK : In particular, we deduce from 3 : if there exists a dominated splitting of f on M, there exists a neighbourhood $\mathcal{V}$ of f (in C^1 topology) such that no g in $\mathcal{V}$ has a periodic orbit $\{p, g(p), \dots, g^{\tau-1}(p)\}$ such that all the eigenvalues of $Dg^\tau(p)$ have a modulus equal to 1.

A question of M. Herman in [5] concerns the converse of this remark. Before giving it, we need some definition (given in [5]) :

DEFINITION : If $f \in \mathrm{Diff}^1_\omega(M)$ leaves invariant a borelian probability measure μ on M, we define :

$$\lambda_+(f,\mu) = \lim_{k\to+\infty} \frac{1}{k}\int_M \log\|Df^k(x)\|_x d\mu(x).$$

We say that f has stably exponents and we write $f \in SE^1_\omega(M)$ if there exist $\delta > 0$ and a neighbourhood $\mathcal{V}$ of f in $\mathrm{Diff}^1_\omega(M)$ (in the C^1-topology) such that for every $g \in \mathcal{V}$ and every borelian probability measure μ invariant by g, we have : $\lambda_+(g,\mu) \geq \delta$.

REMARK : In fact (this fact is reported in [5]), $f \in SE^1_\omega(M)$ is equivalent to :
"there exist $\delta > 0$ and a neighbourhood $\mathcal{V}$ of f in $\mathrm{Diff}^1_\omega(M)$ such that for every $g \in \mathcal{V}$ and every periodic point x (with period τ) of g, we have :

$$\lambda_+(g,\mu_{x,g}) \geq \delta \quad \text{where} \quad \mu_{x,g} = \frac{1}{\tau}\sum_{k=0}^{\tau-1}\delta_{g^k(x)};"$$

(in order to prove this equivalence it is necessary to use the ergodic closing lemma of R. Mañé (see [6] or [1])).
If f is symplectic and has a dominated splitting on M, we have : $f \in SE^1_\omega(M)$.

Question (M. Herman) : Is the set $\{f \in \mathrm{Diff}^1_\omega(M); f$ has a dominated splitting on $M\}$ C^1 dense in $SE^1_\omega(M)$? Note that these two sets are both open in the C^1 topology.

In fact, as it was noticed by M. Herman in an oral communication, in the case of symplectic diffeomorphisms, the good question is :

Question (M. Herman) : Is the set $\{f \in \mathrm{Diff}^1_\omega(M); f$ is partially hyperbolic or hyperbolic on $M\}$ C^1 dense in $SE^1_\omega(M)$?

The notion of partial hyperbolicity we will use is a weaker one than the definitions given in [5] or [8]; we give in fact the definition given in [3] :

DEFINITION : Let $f : M \to M$ be a C^1-diffeomorphism. f is partially hyperbolic if there exists a continuous Df-invariant direct sum decomposition $TM = E^s \oplus E^c \oplus E^u$ where E^s, E^c and E^u are non trivial, some constants $0 < \lambda < 1 < \mu$, $C > 0$ and some riemannian metric $\|.\|$ on TM such that :

- $\forall v \in E^s, \forall n \geq 0, \|Df^n(v)\| \leq C\lambda^n\|v\|$;
- $\forall v \in E^u, \forall n \geq 0, \frac{\mu^n}{C}\|v\| \leq \|Df^n(v)\|$;
- $E^s \prec E^c \prec E^u$.

This means that Df contracts uniformly along E^s and dilates uniformly along E^u, and that at each point of M, if Df contracts (respectively dilates) along E^c, this is less than along E^s (resp. E^u) (but we don't know anything concerning the comparaison of $Df_{|E^s}$ or $Df_{|E^u}$ with $Df_{|E^c}$ at two different points).

Let us define completely elliptic periodic points :

Definition : Let $f : M \to M$ be a C^1-diffeomorphism, and $p \in M$ a periodic point of f with primary period τ. We shall say that p is a *completely elliptic* periodic point of f if the modulus of all the eigenvalues of $Df^\tau(p)$ is 1 and if all these eigenvalues are distinct (in particular, 1 and -1 are not eigenvalues of $Df^\tau(p)$).

Let us notice that if p is a completely elliptic periodic point of $f \in \text{Diff}^1_\omega(M)$, every $g \in \text{Diff}^1_\omega(M)$ C^1-close to f has a completely elliptic periodic point near p. Thus the set of symplectic C^1-diffeomorphisms having a completely elliptic periodic point is open (for the C^1-topology). Of course, it may be non-dense : if for example there exists an Anosov symplectic diffeomorphism.

Theorem 2 *Let (M,ω) be a connected, compact, symplectic 4-dimensional manifold. Then there exists three open sets (for the C^1-topology) U_1, U_2 and U_3 of $\text{Diff}^1_\omega(M)$ such that :*

- *$U_1 \cup U_2 \cup U_3$ is dense in $\text{Diff}^1_\omega(M)$;*
- *$f \in U_1$ if and only if f is Anosov and transitive;*
- *$f \in U_2$ if and only if f is partially hyperbolic; more precisely, there exists a continuous decomposition $TM = E^s \oplus E^c \oplus E^u$ for which f is partially hyperbolic such that : $\dim(E^u) = \dim(E^s) = 1$, $\dim(E^c) = 2$;*
- *$f \in U_3$ if and only if f has a stable completely elliptic periodic point.*

We deduce from this theorem :

Corollary 3 *Let (M,ω) be a connected, compact, symplectic 4-dimensional manifold. Then the set of C^1-diffeomorphisms of M which are hyperbolic or partially hyperbolic is dense in $SE^1_\omega(M)$.*

Remark : The three situations (U_1, U_2 and U_3) may occur. Let us assume that $M = \mathbf{T^4}$ is the 4-torus. If the coordinates are $(\theta_1,\theta_2,\theta_3,\theta_4) \in \mathbf{T}^4$, we consider the symplectic form : $\Omega = d\theta_1 \wedge d\theta_2 + d\theta_3 \wedge d\theta_4$. Then :

- there exist Anosov and transitive diffeomorphisms; for example $f(\theta_1,\theta_2,\theta_3,\theta_4) = (A(\theta_1,\theta_2), A(\theta_3,\theta_4))$ where

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

- there exist partially hyperbolic diffeomorphisms : for example consider $Id \times A$, where A has been defined before;
- there exist diffeomorphisms having a stable completely elliptic periodic point : consider a minimum of a hamiltonian flow and perturb the time-one map to obtain distinct eigenvalues with modulus one.

Of course, there exist manifolds having no Anosov diffeomorphism; does there exist manifolds with no partially hyperbolic diffeomorphisms? See 4.9.b of [5] for a more precise question and the addendum at the end of this introduction for related questions.

Let us notice that $U_1 \cup U_2$ and U_3 are always disjoints, but that sometimes we have : $U_1 \cap U_2 \neq \emptyset$.

Another amusing corollary is :

Corollary 4 *Let (M, ω) be a connected, compact, symplectic 4-dimensional manifold. Let $f \in \text{Diff}^1_\omega(M)$ be a diffeomorphism having a hyperbolic periodic point p such that, if τ is the primary period of p, $Df^\tau(p)$ has a complex eigenvalue (and then all the eigenvalues are complex). Then :*

- *either f is Anosov and transitive;*
- *or in every neighbourhood U of f (in C^1-topology) there exists g having a completely elliptic periodic point.*

Before ending this introduction, let us mention the work [2] of C. Bonatti, L. Diaz and E. Pujals in the case of volume preserving diffeomorphisms (V is a volume element on the compact manifold M) : they prove that if $f \in \text{Diff}^1_V(M)$, there are two possibilities :

- either given any $k \in \mathbf{N}$ there is $g \in \text{Diff}^1_V(M)$ arbitrary C^1-close to f having k periodic orbits whose derivatives are the identity;
- or the manifold M is the union of a finite number of invariant and compact (a priori non disjoint) sets having a dominated splitting.

Let us compare our result with this result : in the case of symplectic diffeomorphisms of 4-dimensional compact and connected manifold our result is more precise in the sense that we obtain a dominated splitting of the whole manifold; and of course if we obtain one completely elliptic periodic point, it is easy to obtain as many completely elliptic periodic points as we want by perturbing the symplectic diffeomorphism (in C^1-topology) : using a generating function, we linearize the diffeomorphism near the periodic point an perturb it in such a way it is a root of he identity and obtain a infinity of (degenerate) periodic point and by another perturbation we can make a finite number of these periodic points completely elliptic.

Moreover, we obtain not only a dominated splitting, but the fact that the diffeomorphism is hyperbolic or partially hyperbolic. Let us recall that using a weaker definition of partial hyperbolicity (with two bundles), L. Diaz, E. Pujals and R. Ures proved in [3] that every robustly transitive diffeomorphism of a 3-dimensional compact and connected manifold is partially hyperbolic.
Let us give the structure of the proof of the theorem 2 :

- in section 2, we shall study 4-dimensional symplectic vector spaces, and more precisely planes in these spaces;
- in section 3 we shall explain how we can reduce the problem to the study of an elliptic periodic linear system;
- in section 4 we shall study this elliptic periodic linear system.

Addendum : after I have written this article, M. Herman introduced the notion of K.A.M. manifold (conference in June 2000, séminaire de système dynamique at University Paris 7) : a K.A.M. manifold is a symplectic manifold which carries no partially hyperbolic or Anosov symplectic diffeomorphism. Using Chern classes (joint work with D. Bennequin), he proved that a lot of manifolds are K.A.M. manifolds; for example, $\mathbf{P}^1(\mathbb{C}) \times \mathbf{P}^1(\mathbb{C})$ and $\mathbf{P}^2(\mathbb{C})$ are K.A.M. manifolds; therefore, by theorem 2, there exists for these two manifolds an open dense subset U_3 of $\text{Diff}^1_\omega(M)$ such that every $f \in U_3$ has a completely elliptic periodic point.

2 4-dimensional symplectic vector spaces.

Let us recall :
DEFINITION : Let E be a 4-dimensional real vector space. Let $\omega : E \times E \to \mathbb{R}$ be a skew symmetric nondegenerate bilinear 2-form. Then we say that ω is a symplectic form on E and that (E, ω) is a symplectic vector space.

When (E, ω) is a 4-dimensional symplectic vector space, there exists a linear map $A : E \to \mathbb{R}^4$ which maps ω on the standard symplectic form Ω_0 defined on $\mathbb{R}^4$ by :

$$\Omega_0((x_1, x_2, y_1, y_2), (x'_1, x'_2, y'_1, y'_2)) = x_1 y'_1 - x'_1 y_1 + x_2 y'_2 - x'_2 y_2;$$

this is written : $\Omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Therefore, from now, we shall work in $(\mathbb{R}^4, \Omega_0)$. Let us introduce some notations :

- $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$;
- (e_1, e_2, e_3, e_4) is the canonical basis of $\mathbb{R}^4$. Let us notice that the matrix of Ω_0 in the basis (e_1, e_2, e_3, e_4) is :
$$\Omega_0 = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}.$$

Now, imagine that we want to write $\omega = \Omega_0$ in a orthonormal basis whose two first vectors are in a fixed plane. Then we have :

Proposition 5 *Let $\mathcal{P}$ be a plane of $\mathbb{R}^4$. There exists a unique $a \in [0, 1]$, there exists an orthogonal matrix $P \in O(4)$ such that :*

(1) *$P(\mathbb{R}e_1 + \mathbb{R}e_2) = \mathcal{P}$ (and then $P(\mathbb{R}e_1 + \mathbb{R}e_2)^\perp) = \mathcal{P}^\perp$) ;*

(2) *if $\Omega := {}^tP\Omega_0 P$ (then Ω is the matrix of ω in the basis (Pe_1, Pe_2, Pe_3, Pe_4)), we have*
$$\Omega = \begin{pmatrix} aJ & \sqrt{1-a^2}\mathbf{1} \\ -\sqrt{1-a^2}\mathbf{1} & -aJ \end{pmatrix}.$$

Proof of the proposition 5 : If P is a 4-dimensional square matrix, we write :

$$P = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}$$

where P_1, P_2, P_3, P_4 are 2-dimensional square matrices. The following lemma is easy to be proved :

Lemma 6 *P is orthogonal if and only if :* $\begin{cases} {}^tP_1P_1 + {}^tP_3P_3 = \mathbf{1} \\ {}^tP_2P_2 + {}^tP_4P_4 = \mathbf{1} \\ {}^tP_1P_2 + {}^tP_3P_4 = \mathbf{0} \end{cases}$.

We know that there exists an orthogonal matrix (we call it Q) such that : $Q(\mathbb{R}e_1 + \mathbb{R}e_2) = \mathcal{P}$. Then we define : $\Omega_1 = {}^tQ\Omega_0 Q$. This matrix is orthogonal and antisymmetric. Therefore, there exist a 2-dimensional matrix A, two reals a_1 and a_2 such that :

$$\Omega_1 = \begin{pmatrix} a_1 J & A \\ -{}^tA & a_2 J \end{pmatrix}$$

(here we used the fact that each antisymmetric 2-dimensional matrix is a multiple of J).

Moreover, if $R = \begin{pmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$, then R is orthogonal, satisfies $QR(\mathbb{R}e_1 + \mathbb{R}e_2) = \mathcal{P}$ and : ${}^t(QR)\Omega_0 QR = {}^tR\Omega R = \begin{pmatrix} -a_1 J & B \\ -{}^tB & a_2 J \end{pmatrix}$ where B is a 2 -dimensional matrix. Thus we can assume that $a_1 \geq 0$.

Using lemma 6, we write that Ω_1 is orthogonal ; this is equivalent to :

(1) $A \cdot {}^tA = (1 - a_1^2)\mathbf{1}$;

(2) ${}^tA \cdot A = (1 - a_2^2)\mathbf{1}$;

(3) $a_1 JA + a_2 AJ = 0$.

Let us recall that we have the equivalence :

$${}^tMM = 0 \Longleftrightarrow M = 0.$$

Thus we have to consider two cases :
First case : $A = 0$. Then $1 - a_1^2 = 1 - a_2^2 = 0$ and thus $a_1 = 1$ and $a_2 \in \{-1, 1\}$. If $a_2 = 1$, we notice that if $S = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix}$, then : ${}^tS\Omega_1 S = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}$. Thus we may assume that

$$\Omega = \Omega_1 = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}.$$

Second case : $A \neq 0$. Thus $1 - a_1^2 \neq 0$ and $1 - a_2^2 \neq 0$. As ${}^tA \cdot A = (1 - a_2^2)\mathbf{1}$, A est invertible. We can write : ${}^tAA = {}^tA(A{}^tA){}^tA^{-1}$, thus ${}^tAA$ and $A{}^tA$ have the same eigenvalues : $1 - a_1^2 = 1 - a_2^2$, i.e. $a_2 = \pm a_1$. Moreover, as ${}^tAA = (1 - a^2)\mathbf{1}$, the matrix $\frac{1}{\sqrt{1-a^2}}A =: S$ is orthogonal. Let us define :

$$P_3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & {}^tS \end{pmatrix};$$

then P_3 is orthogonal, $P_3(\mathbb{R}e_1 + \mathbb{R}e_2) = \mathbb{R}e_1 + \mathbb{R}e_2$ and

$$\Omega_2 = {}^tP_3\Omega_1 P_3 = \begin{pmatrix} aJ & \sqrt{1-a^2}\mathbf{1} \\ -\sqrt{1-a^2}\mathbf{1} & a_2'J \end{pmatrix}.$$

Let us write that Ω_2 is orthogonal (see lemma 6) :

(1) $a^2 + 1 - a^2 = 1$;

(2) $a^2 + 1 - a^2 = 1$;

(3) $-aJ\sqrt{1-a^2} - \sqrt{1-a^2}a_2'J = 0$, i.e. $a_2' = -a$.

Then we obtain :

$$\Omega = \Omega_2 = \begin{pmatrix} aJ & \sqrt{1-a^2}\mathbf{1} \\ -\sqrt{1-a^2}\mathbf{1} & -aJ \end{pmatrix}.$$

Let us prove now that a is unique. In fact, we easily verify that :

$$a = \|\omega_{|\mathcal{P}}\| = \sup\{|\omega(x,y)|, (x,y) \in \mathcal{P}^2, \|x\| = \|y\| = 1\}.$$

■

Definition : Let E and F be two non null vector subspaces of $\mathbb{R}^4$. Then :

- $\delta(E,F) = \mathrm{Min}\{\|x-y\|; (x,y) \in E \times F, \|x\| = \|y\| = 1\}$ is the “distance” between E and F (in fact δ is not a distance) ; let us notice that we always have :

$$0 \leq \delta(E,F) \leq \sqrt{2};$$

- we define the angle between E and F as being :

$$\angle(E,F) = 2\arcsin\frac{\delta(E,F)}{2};$$

let us notice that we always have :

$$\angle(E,F) \in \left[0, \frac{\Pi}{2}\right].$$

Remark : We can notice that :

- $\angle(E,F) = 0$ iff $E \cap F \neq \{0\}$;
- $\angle(E,F) = \frac{\Pi}{2}$ iff E and F are orthogonal.

Corollary 7 *Let $\mathcal{P}$ be a plane of $\mathbb{R}^4$. There exists an orthonormal basis (f_1, f_2, f_3, f_4) of $\mathbb{R}^4$ such that $\mathcal{P} = \mathbb{R}f_1 + \mathbb{R}f_2$ (and thus $\mathcal{P}^\perp = \mathbb{R}f_3 + \mathbb{R}f_4$) and such that, if we write : $x = x_1 f_1 + x_2 f_2$ and $y = y_1 f_1 + y_2 f_2$, the equation of $\mathcal{P}^{\perp\omega}$ (the orthogonal subspace of $\mathcal{P}$ for ω) is :*

$$aJx + \sqrt{1-a^2}y = 0\,.$$

Proof of the corollary 7 : We use the notations of the proposition 5 and define :

$$(f_1, f_2, f_3, f_4) = (Pe_1, Pe_2, Pe_3, Pe_4);$$

then we have :

$$\forall (x,y) \in \mathbb{R}^2, (\mathbf{1}, 0) \begin{pmatrix} aJ & \sqrt{1-a^2}\mathbf{1} \\ -\sqrt{1-a^2}\mathbf{1} & -aJ \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = aJx + \sqrt{1-a^2}y\,.$$

■

Corollary 8 *Let $\mathcal{P}$ be a plane of $\mathbb{R}^4$. Then, for all lines D_1 and D_2 such that $D_1 \subset \mathcal{P}$ and $D_2 \subset \mathcal{P}^{\perp\omega}$, we have :*

$$\angle(\mathcal{P}, \mathcal{P}^{\perp\omega}) = \angle(D_1, \mathcal{P}^{\perp\omega}) = \angle(\mathcal{P}, D_2)\,.$$

Proof of corollary 8 : the result comes from the equation of $\mathcal{P}^{\perp\omega}$ given in corollary 7 :

$$aJx + \sqrt{1-a^2}y = 0\,.$$

Indeed, this equation implies :
$\forall x \in \mathbb{R}^2, \|x\| = 1 \Rightarrow$

$$\begin{aligned} d(x_1 f_1 + x_2 f_2, \{y \in \mathcal{P}^{\perp\omega}, \|y\| = 1\}) &= d\big((x,0), (\sqrt{1-a^2}x, -aJx)\big) \\ &= \sqrt{(1-\sqrt{1-a^2})^2 + a^2} \\ &= \sqrt{2(1-\sqrt{1-a^2})}\,. \end{aligned}$$

Thus

$$\forall x \in \mathcal{P}, \|x\| = 1 \Rightarrow \delta(\mathbb{R}x, \mathcal{P}^{\perp\omega}) = \delta(\mathcal{P}, \mathcal{P}^{\perp\omega}) = \sqrt{2(1-\sqrt{1-a^2})}\,.$$

To prove the last equality, we notice that :

$$(\mathcal{P}^{\perp\omega})^{\perp\omega} = \mathcal{P}.$$

■

Remark :

(1) we have proved in fact that :

$$\delta(\mathcal{P},\mathcal{P}^{\perp\omega}) = \sqrt{2(1-\sqrt{1-a^2})};$$

and then :

$$\angle(\mathcal{P},\mathcal{P}^{\perp\omega}) = 2\arcsin\sqrt{\frac{1-\sqrt{1-a^2}}{2}}.$$

(2) Let us detail some particular cases :

- we have :

$$\begin{aligned}\mathcal{P} \text{ is lagrangian} \quad &\text{iff} \quad \angle(\mathcal{P},\mathcal{P}^{\perp\omega}) = 0\\ &\text{iff} \quad a = \|\omega_{|\mathcal{P}}\| = 0\,;\end{aligned}$$

- we have :

$$\angle(\mathcal{P},\mathcal{P}^{\perp\omega}) = \frac{\Pi}{2} \quad \text{iff} \quad a = \|\omega_{|\mathcal{P}}\| = 1\,.$$

(3) Let us explain what si the meaning of the last corollary : if $\mathcal{P}$ is a plane of $\mathbb{R}^4$, then the angle between $\mathcal{P}$ and $\mathcal{P}^{\perp\omega}$ is in a certain sense “uniform” : it doesn’t depend on the direction of $\mathcal{P}$ we consider. This is a remarkable property, which is not true if we consider any pair of planes (E,F).

3 We reduce the problem to the study of an elliptic periodic linear system.

Let (M, ω) be a connected and compact symplectic 4-dimensional manifold.
REMARK : The set U_1 of Anosov symplectic C^1-diffeomorphisms of M is open in $\text{Diff}^1_\omega(M)$. Moreover, if $f \in U_1$, f is an Anosov diffeomorphism of a connected manifold whose non-wandering set is the whole manifold; therefore f is an axiom A diffeomorphism, M is a basic set of f and then f is transitive.
Thus U_1 is also the set of transitive Anosov C^1-diffeomorphisms of M.

From now, we shall interest ourselves in describing diffeomorphisms $f \in \text{Diff}^1_\omega(M)$ which are not Anosov. We shall use the following result, proved in [7] :

Theorem 9 (S. Newhouse) *Let (M, ω) be a compact and connected symplectic manifold. There exists a residual subset $\mathcal{D} \subset \text{Diff}^1_\omega(M)$ such that if $f \in \mathcal{D}$, either f is Anosov or the elliptic periodic points of f are dense in M.*

Using the notation of theorem 2, we shall assume : $f \in \mathcal{D} \backslash (\overline{U}_1 \cup \overline{U}_3)$. The set $\mathcal{E}$ of elliptic periodic points of f is then dense in M. Moreover, as $f \notin \overline{U}_3$, all the elliptic periodic points of f are partially hyperbolic. We can define a splitting of $TM_{|\mathcal{E}}$:

$$TM_{|\mathcal{E}} = S \oplus E \oplus U$$

where, if $x \in \mathcal{E}$ and τ is the primary period of x, S_x is the stable manifold of $Df^\tau(x)$, U_x its unstable manifold and E_x its center manifold.

Of course, M is endowed with a riemannian metric. But we want to consider in every fiber T_xM of TM a very particular scalar product. We shall named it $\langle \cdot, \cdot \rangle_x$ and it is such that there exists a linear operator : $J_x = T_xM \to T_xM$ such that $J_x^2 = -\text{Id}_{T_xM}$ (thus J_x is an almost complex structure for $\langle \cdot, \cdot \rangle_x$) and : $\forall (v, w) \in (T_xM)^2$, $\omega(x)(v, w) = \langle v, J_x w \rangle_x$. $(x \to J_x)$ and $(x \to \langle \cdot, \cdot \rangle_x)$ are not assumed to be continuous, but we assume that they are upper and lower bounded on M by two strictly positive constants. Thus the norm defined on TM by the initial riemannian structure and $\langle \cdot, \cdot \rangle_x^{1/2}$ are uniformly equivalent on M. From now, when we shall write $\| \cdot \|_x$, it will be $\langle \cdot, \cdot \rangle_x^{1/2}$. Let us notice that J_x is an isometry for $\| \cdot \|_x$.

Proposition 10 *Let $f \in \mathcal{D} \backslash (\overline{U}_1 \cup \overline{U}_3)$ be such that we have on $\mathcal{E}$:*

$$S \prec E \prec U \, .$$

Then f is partially hyperbolic on M, and the corresponding splitting restricted to $\mathcal{E}$ is $S \oplus E \oplus U$.

Proof of the proposition 10 : We know that there is an unique extension of the dominated splitting $S \prec E \prec U$ to $\overline{\mathcal{E}} = M$. This extension will be called again $S \oplus E \oplus U$. Thus there exists $N \geq 1$ such that :

$$\forall x \in M, \begin{cases} \|Df^N(x)_{|S}\|_x \leq \frac{1}{2}\|(Df^N(x))^{-1}_{|E}\|_x^{-1} \\ \|Df^N(x)^{-1}_{|U}\|_x \leq \frac{1}{2}\|Df^N(x)_{|E}\|_x^{-1} \end{cases}$$

As we have a dominated decomposition, we know that $\inf_{x\in\mathcal{E}} \angle(S_x, E_x) > 0$ and therefore : $\inf_{x\in\mathcal{E}} \|\omega_{|E_x\times E_x}\| > 0$. Therefore there exists $\mathcal{J}_x : E_x \to E_x$ such that $\mathcal{J}_x^2 = -Id_{E_x}$ and a scalar product $(.|.)_x$ such that : $\forall (v,w) \in E_x^2, \omega(x)(v,w) = (v|\mathcal{J}_x w)$. $(x \to J_x)$ and $(x \to (.|.)_x)$ are not assumed to be continuous but are upper and lower bounded by strictly positive constants. Let us call p the norm associated to $(.|.)_x$; we have :

$$\forall v \in E_x, \alpha p(v) \leq \|v\|_x \leq \beta p(v).$$

Moreover $\mathcal{J}_x$ is an isometry for p.

But $Df^n(x) = E_x \to E_{f^n x}$ is symplectic, i.e.

$$\forall (v,w) \in (T_x M)^2, \omega(x)(v,w) = \omega(f^n x)(Df^n(x)v, Df^n(x)w)\,.$$

In particular :

$$\begin{aligned} \forall v \in E_x, p(v)^2 &= \omega(x)(v, -\mathcal{J}_x v) \\ &= \omega(f^n x)(Df^n(x)v, -Df^n(x)\mathcal{J}_x v) \\ &= (Df^n(x)v| - \mathcal{J}_{f^n x} Df^n(x)\mathcal{J}_x v)\,. \end{aligned}$$

Therefore, because $\mathcal{J}_x$ is an isometry for p :

$$\forall v \in E_x, p(v)^2 \leq p\left(Df^n(x)_{|E_x}\right)^2\ p(v)^2\,.$$

Thus :

$$p(Df^n(x)_{|E_x}) \geq 1$$

Let us choose $k \in \mathbf{N}$ such that $2^k > \frac{2\beta}{\alpha}$. Then we have :

$$\|Df^{kN}(x)^{-1}_{|U}\|_x \leq \frac{1}{2^k}\|Df^{kN}(x)_{|E}\|_x^{-1} \leq \frac{1}{2^k}\frac{\beta}{\alpha}p(Df^{kN}(x)_{|E})^{-1} \leq \frac{1}{2}$$

and so, if we replace N by kN, we have :

$$\|Df^N(x)^{-1}_{|U}\|_x \leq \frac{1}{2}$$

and by the same method :

$$\|Df^N(x)_{|S}\|_x \le \frac{1}{2}$$

i.e.

$$\forall x \in M, \forall v \in S_x, \|Df^N(x)\cdot v\|_x \le \frac{1}{2}\|v\|_x\,.$$

Then we have :

$$\forall x \in M, \forall v \in S_x, \forall k \in \{0,\dots,N-1\}, \forall K \in \mathbb{N}\,,$$
$$\|Df^{KN+k}(x)\cdot v\|_x \le \left(\left(\frac{1}{2}\right)^{1/N}\right)^{KN+k} 2\max_{y\in M}\{1, \|Df(y)\|_y, \dots, \|Df^{N-1}(y)\|_y\}\|v\|_x$$

i.e. there exist $C > 0$ and $\lambda > 0$ such that :

$$\forall x \in M, \forall v \in S_x$$
$$\|Df^n(x)v\|_x \le C\lambda^n\|v\|_x\,.$$

Of course, we can replace $\|\cdot\|_x$ by the initial riemannian norm : only the constant C changes.

To prove the inequality concerning U, we change f into f^{-1}.

■

A consequence of this last proposition is : if $f \in \mathcal{D}\backslash(\overline{U_1}\cup\overline{U_3})$ is not partially hyperbolic, then S (on $\mathcal{E}$) is not dominated by E or E (on $\mathcal{E}$) is not dominated by U. As we can change f into f^{-1}, we shall assume that S is not dominated by E.

Let us assume that we have proved that :

"for all $f \in \mathcal{D}\backslash(\overline{U_1}\cup\overline{U_3})$, if S is not dominated on $\mathcal{E}$ by E, then $f \in \overline{U}_3$ "

this gives a contradiction and implies that every $f \in \mathcal{D}\backslash(\overline{U_1}\cup\overline{U_3})$ is partially hyperbolic. The set U_2 of partially hyperbolic symplectic diffeomorphisms of M is open (in C^1-topology) and we conclude that $\overline{U_1}\cup\overline{U_2}\cup\overline{U_3} = \mathrm{Diff}^1_\omega(M)$, therefore the theorem 2 is proved.

4 Study of an elliptic periodic linear system.

As it was explained at the end of the previous section, we only have to prove :

Proposition 11 *Let $f \in \mathcal{D} \backslash (\overline{U_1} \cup \overline{U_3})$ be such that S is not dominated by E on $\mathcal{E}$; then $f \in \overline{U}_3$.*

Thus let us assume that $f \in \mathcal{D} \setminus (\overline{U_1} \cup \overline{U_3})$ and that S is not dominated by E on $\mathcal{E}$. There are two cases :

(1) There exists $\alpha > 0$ such that :

$$\forall x \in \mathcal{E}, \angle(S_x, E_x) \geq \alpha;$$

(2) $\inf_{x \in \mathcal{E}} \angle(S_x, E_x) = 0$

where each fiber $T_x M$ $(x \in \mathcal{E})$ is endowed with the euclidean norm $\| \cdot \|_x$ defined in section 3 and the angle is defined as in section 2. Moreover, we shall fix in each fiber $T_x M$ (here $x \in M$) an orthonormal (for $\langle \cdot, \rangle_x$) and symplectic basis $(e_{1,x}, e_{2,x}, e_{3,x}, e_{4,x})$. Then we have :

$$\begin{aligned} \forall x \in \mathcal{E}, \omega(x)(e_{1,x}, e_{3,x}) &= \omega(x)(e_{2,x}, e_{4,x}) = 1 \\ \omega(x)(e_{1,x}, e_{2,x}) &= \omega(x)(e_{1,x}, e_{4,x}) = \omega(x)(e_{2,x}, e_{3,x}) \\ &= \omega(x)(e_{3,x}, e_{4,x}) = 0. \end{aligned}$$

In fact in this section we shall only study the linear periodic system $(Df(x))_{x \in \mathcal{E}}$ of $TM_{|\mathcal{E}}$ (see [2] for a detailed definition). A very useful lemma to transform perturbations of this linear periodic system into perturbations of the diffeomorphim f is the so-called conservative Franks' lemma (see [2]) :

Lemma 12 (Franks) *Let $U \subset \text{Diff}^1_\omega(M)$ be a neighbourhood (in C^1-topology) of f given. The there exists $\varepsilon > 0$ such that :*

For every finite f-invariant subset F of M, for every symplectic ε-perturbation B of $Df_{|TM_{|F}}$ along F, for every neighbourhood V of F, there exist $g \in U$ such that :

- $f_{|F} = g_{|F}$;
- $g_{|M \backslash V} = f_{|M \backslash V}$;
- $\forall x \in F, Dg(x) = B_x$.

4.1 The first case.

We assume that there exists $\alpha > 0$ such that :

$$\forall x \in \mathcal{E} \ , \ \angle(S_x, E_x) \geq \alpha$$

and we fix a neighbourhood U of f (in C^1-topology). We shall prove :
$(*)$ "for every $\varepsilon > 0$, there exists $g \in U$ having an elliptic $\times$ hyperbolic periodic point x such that :

$$\angle(S_x, E_x) < \varepsilon \, . \text{"}$$

If $U \cap (\overline{U}_1 \cup \overline{U}_3) = \emptyset$ (we may assume that because $f \notin \overline{U}_1 \cup \overline{U}_3$), we have this result near every $g \in \mathcal{D} \cap U$ and for every $\varepsilon = \frac{1}{n}$ (if $\inf_{x \in \mathcal{E}} < (S_x, E_x) \neq 0$, it comes from $(*)$ and if not the result is obvious). Thus doing a countable intersection and using Baire theorem, we obtain :
$(**)$ "there exists $g \in U \cap \mathcal{D} \backslash (\overline{U}_1 \cup \overline{U}_3)$ such that :

$$\inf_{x \in \mathcal{E}} \angle(E_x, S_x) = 0 \, . \text{"}$$

And then we only have to study the second case (next subsection) :

$$\inf_{x \in \mathcal{E}} \angle(E_x, S_x) = 0 \, .$$

But before that, let us prove $(*)$.

Proof of $(*)$ *:* we assume that S is not dominated by E. Then, for every $n \in \mathbb{N}$, there exists $x_n \in \mathcal{E}$ such that :

$$\|Df^n(x_n)_{|S_{x_n}}\| \cdot \|Df^{-n}(f^n x_n)_{|E_{f^n x_n}}\| > \frac{1}{2} \, .$$

Let us fix $n \in \mathbb{N}^*$ (we shall explain later how it is chosen). Using generating functions, we can make a small perturbation $\widetilde{f}$ of f (as small as we want in the C^1-topology) such that :

(1) x_n (periodic with primary period named τ) has the same orbit $O(x_n)$ for $\widetilde{f}$ than for f ;

(2) $\forall x \in O(x_n)$, $D\widetilde{f}(x)_{|U_x + S_x} = Df(x)_{|U_x + S_x}$ and $D\widetilde{f}^\tau(x_n)_{|E_n}$ is conjugated to a rational rotation whose order μ is even and bigger than n ;

(3) $\widetilde{f}$ is linear (in a charts) in a small neighbourhood of the orbit of x_n.

If now $y \in E_n$ is close to x_n but different from x_n, y is periodic (for $\widetilde{f}$) with primary period $\mu\tau \geq n$, and $D\widetilde{f}^{\mu\tau}(y)_{|E_y} = \mathrm{Id}_{E_y}$. Let us notice that in the considered chart, we can write : $E_y = E_{x_n}$, $S_{x_n} = S_y$, $U_{x_n} = U_y$.

Moreover, as for f, there exists $s \in S_y$ and $e \in E_y$ such that :

- $\|s\|_y = \|e_y\| = 1$;
- $\|D\widetilde{f}^n(y) \cdot s\| > \frac{1}{2}\|D\widetilde{f}^n(y) \cdot e\|$.

Let us explain now how we have chosen n. We use a lightly modified version of lemma 3.4 in [2] :

Lemma 13 *for any $\varepsilon_1 > 0$, $\varepsilon > 0$ and $K > 0$, there exists $n \in \mathbb{N}^*$ with the following property :*

for every N-uple $(A_1, \dots, A_N)$ of diagonal matrices of $\mathrm{GL}_+(2,\mathbb{R})$ bounded by K (with $N \geq n$) such that $A_i = \begin{pmatrix} a_i & 0 \\ 0 & b_i \end{pmatrix} = |a_1 \dots a_N| \leq |b_1 \dots b_N|$, and such that $\left|\prod_{i=1}^n a_i\right| > \dfrac{1}{2}\left|\prod_{i=1}^n b_i\right|$, there exists an ε_1-perturbation $(\widetilde{A}_1, \dots, \widetilde{A}_N)$ of $(A_1, \dots, A_N)$ such that the angle between the eigenspaces of $\widetilde{A}_N \cdot \dots \cdot \widetilde{A}_1$ is less than ε.

Let us notice that the last case of the proof of lemma 3-4 in [2] $(n > N)$ is not considered here, because it is useless.

We shall work in the following planes : $\mathcal{P} = \mathbb{R}e \oplus \mathbb{R}s \subset T_yM$, $D\widetilde{f}(y)(\mathcal{P}), \dots, D\widetilde{f}^{\mu\tau-1}(y)(\mathcal{P})$ and we shall use the following basis in $\mathcal{P}_k = D\widetilde{f}^k(y)(\mathcal{P})$ (where $0 \leq k \leq \mu\tau - 1$) :

$$(e_k, s_k) = \left(\frac{D\widetilde{f}^k(y)e}{\|D\widetilde{f}^k(y)e\|}, \frac{D\widetilde{f}^k(y)s}{\|D\widetilde{f}^k(y)s\|} \right).$$

Of course, this basis is not orthogonal, but we know that :

$$\angle(\mathbb{R}e_k, \mathbb{R}s_k) \geq \alpha$$

thus by using a bounded matrix P (with P^{-1} bounded), we may assume that (e_k, s_k) is orthonormal. We consider the matrices $A_0, \dots, A_{\mu\tau}$ of $D\widetilde{f}(y)_{|\mathcal{P}}, \dots, D\widetilde{f}^{\mu\tau}(y)_{|\mathcal{P}}$ in this basis. They are in $\mathrm{GL}_+(2,\mathbb{R})$. They satisfy all the assumptions of the previous lemma and we can find $\widetilde{A}_1, \dots, \widetilde{A}_{\mu\tau}$ which satisfy the conclusion of the lemma. Moreover, it is easy to see (by reading carefully the original proof of the lemma) that we can assume that one of the eigenvalues of $(\widetilde{A}_{\mu\tau} \dots \widetilde{A}_1)$ is 1 and the other is positive and strictly less than 1.

Now we remark that every $\mathcal{P}_k$ is lagrangian. We can find in every space $T_{\widetilde{f}^k y}M$ a symplectic basis (e, s, e', s') such that, if P is the matrix which sends (e_1, e_2, e_3, e_4) on (e, s, e', s'), $\|P\|$ and $\|P^{-1}\|$ are bounded by an a priori constant (because $\angle(E, S \oplus U) \geq \alpha$ and ω in bounded); thus we have :

$$\begin{cases} \omega(e, e') = \omega(s, s') = 1 \\ \omega(e, s) = \omega(e, s') = \omega(e', s) = \omega(e', s') = 0. \end{cases}$$

Thus to obtain a small perturbation of $D\widetilde{f}(y), \dots, D\widetilde{f}(f^{\mu\tau-1}y)$, it suffices to obtain a small perturbation of $PD\widetilde{f}(y)P^{-1}, \dots, PD\widetilde{f}(f^{\mu\tau-1}y)P^{-1}$. But if we define :

$$M_k = PD\widetilde{f}(f^k y)P^{-1}$$

we have :

$$M_k = \begin{pmatrix} A_k & B \\ 0 & C \end{pmatrix}.$$

P being symplectic, M_k is symplectic, and we see easily that this means :

$$M_k = \begin{pmatrix} A_k & A_k S_k \\ 0 & {}^t A_k^{-1} \end{pmatrix}$$

where S_k is a symmetric matrix.

We can perturb this matrix in another one :

$$\widetilde{M_k} = \begin{pmatrix} \widetilde{A}_k & \widetilde{A}_k S_k \\ 0 & {}^t \widetilde{A}_k^{-1} \end{pmatrix}$$

which is symplectic and close to M_k because P and P^{-1} are bounded by a priori constants.

Using Franks' lemma, we perturb $\widetilde{f}$ in $\widetilde{g}$ such that the orbit of y is the same for $\widetilde{f}$ and $\widetilde{g}$ and such that :

$$\forall k \in 1, \dots, \mu\tau - 1 \ , \ D\widetilde{g}(\widetilde{g}^k y) = P^{-1} \widetilde{M_k} P \, .$$

Then we have :

$$PD\widetilde{g}^{\mu\tau}(y)P^{-1} = \begin{pmatrix} \widetilde{A}_{\mu\tau} \cdot \ldots \cdot \widetilde{A}_1 & X \\ 0 & {}^t(\widetilde{A}_{\mu\tau} \ldots \widetilde{A}_1)^{-1} \end{pmatrix}$$

and thus :

- y is a elliptic $\times$ hyperbolic periodic point of $\widetilde{g}$ with period $\mu\tau$;
- for $\widetilde{g}$, the angle between E_y and $S_y + U_y$ is less than ε.

■

4.2 The second case.

We assume that :

$$\inf_{x \in \mathcal{E}} \angle(S_x, E_x) = 0 \, .$$

We have seen that this can be written too :

$$\inf_{x\in\mathcal{E}} \angle(S_x + U_x, E_x) = 0\,.$$

Then for every $a_0 \in]0,1[$, there exists $x \in \mathcal{E}$ such that : $\|\omega_{|E_x}\| = a \leq a_0$. For such a x, there exists an orthonormal basis $(e_{1,x}, e_{2,x}, e_{3,x}, e_{4,x})$ of T_xM, such that $E_x = \mathbb{R}e_{1,x} \oplus \mathbb{R}e_{2,x}$ and such that the matrix of ω in this basis is :

$$\Omega = \begin{pmatrix} aJ & \sqrt{1-a^2}\mathbf{1} \\ -\sqrt{1-a^2}\mathbf{1} & -aJ \end{pmatrix}.$$

Lemma 14 *If $f \in \mathcal{D}\backslash(\overline{U_1} \cup \overline{U_3})$ is such that $\inf_{x\in\mathcal{E}} \angle(S_x, U_x) = 0$, then $f \in \overline{U_3}$.*

Proof of lemma 14 : Let f be such a diffeomorphism. Let U be a C^1-neighbourhood of f. Then, using Franks' lemma, we may choose an ε. Let $x \in \mathcal{E}$ be such that $\angle(S_x, U_x) < \varepsilon$. There exists a small perturbation $\tilde{f}$ of f such that :

- the orbit of x under $\tilde{f}$ and f are the same;
- for all y on the orbit of x :

$$D\tilde{f}(y)_{|U_y+S_y} = Df(y)_{|U_y+S_y};$$

- $D\tilde{f}^t(x)_{|E_x}$ is conjugated to a rational rotation (t is the primary period of x);
- $\tilde{f}^t$ is linear near x (in a chart).

Then, using the same argument than in section 4.1, we create a periodic point y (with period τ) of $\tilde{f}$ such that :

- $\angle(S_y, U_y) < \varepsilon$;
- $D\tilde{f}^\tau(y)_{|E_y} = Id_{E_y}$.

The matrix of $Df^\tau(y)$ in the basis $(e_{1,y}, e_{2,y}, e_{3,y}, e_{4,y})$ is :

$$M = \begin{pmatrix} \mathbf{1} & C \\ \mathbf{0} & H \end{pmatrix}.$$

where the eigenvalues of H are $\lambda > 1$ and $\frac{1}{\lambda}$.
If the angle between U_y and S_y is less than ε, the lemma 3.2 of [2] implies that there exists an $\theta \in [-\varepsilon, \varepsilon]$ such that $R_\theta H$ has two eigenvalues with modulus equal to 1 (where R_θ is the usual matrix of the rotation of angle θ). Let us define :

$$P = \begin{pmatrix} R_\theta & \mathbf{0} \\ \mathbf{0} & R_\theta \end{pmatrix}.$$

This matrix is symplectic for Ω, close to identity, and then, using Franks'lemma, we can perturb $\tilde{f}$ in $\bar{f}$ such that the orbit of y under $\bar{f}$ is the orbit of y under $\tilde{f}$ and such that in the basis $(e_{1,y}, e_{2,y}, e_{3,y}, e_{4,y})$, the matrix of $D\bar{f}^\tau(y)$ is :

$$PM = \begin{pmatrix} R_\theta & R_\theta . C \\ \mathbf{0} & R_\theta H \end{pmatrix} .$$

thus y is completely elliptic.

■

Let us notice that we can do this perturbation even if a is "big" (we have in fact done that in the previous subsection), thus from now, we will then assume that :

$$\inf_{x \in \mathcal{E}} \angle(S_x, U_x) > 0 \, .$$

Let now $x \in \mathcal{E}$. Doing the same argument than in the proof of the previous lemma we can replace x by y with primary period τ such that :

- $\angle(E_x, S_x) = \angle(E_y, S_y)$;
- $\angle(S_x, U_x) = \angle(S_y, U_y)$;
- $Df^\tau(y)_{|E_y} = Id_{E_y}$.

Because $\inf_{x \in \mathcal{E}} \angle(S_x, U_x) > 0$, there exists for every $x \in \mathcal{E}$ a matrix $P = P_x$ such that P_x and P_x^{-1} are uniformly bounded which satisfies :

$$P^{-1} H P = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} ;$$

where $\lambda > 1$. Let us explain what happens for y :

Lemma 15 *For every such y, there exists a matrix Q_y which is symplectic (for Ω), such that $Q = Q_y$ and Q_y^{-1} are uniformly bounded and such that the matrix of $Df^\tau(y)$ in the basis $Q(e_{1,y}, e_{2,y}, e_{3,y}, e_{4,y}) = (e_1, e_2, e_3, e_4)$ is :*

$$M' = \begin{pmatrix} \mathbf{1} & \frac{\sqrt{1-a^2}}{a} \begin{pmatrix} 0 & \frac{1}{\lambda} - 1 \\ 1 - \lambda & 0 \end{pmatrix} \\ \mathbf{0} & \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \end{pmatrix} .$$

Moreover, if $\mathcal{P}_1 = \mathbb{R}e_1 \oplus \mathbb{R}e_4$ and $\mathcal{P}_2 = \mathbb{R}e_2 \oplus \mathbb{R}e_3$, $\mathcal{P}_1$ and $\mathcal{P}_2$ are two lagrangian (for Ω) planes stable by $Df^\tau(y)$ and we have :

$$\mathcal{P}_1 = \mathbb{R}e_1 \oplus S_y ; \mathcal{P}_2 = \mathbb{R}e_2 \oplus U_y .$$

Proof of lemma 15 : We define : $P = P_y$ is such that :

$$P^{-1}HP = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix};$$

Moreover, we may choose P such that : $\det P = 1$. Let us define :

$$Q = \begin{pmatrix} {}^tP^{-1} & \mathbf{0} \\ \mathbf{0} & P \end{pmatrix};$$

then a direct computation prove that Q is symplectic for Ω. Moreover Q and Q^{-1} are uniformly bounded and :

$$M' = Q^{-1}MQ = \begin{pmatrix} \mathbf{1} & B \\ \mathbf{0} & \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \end{pmatrix}.$$

If we write that $Q^{-1}MQ$ is symplectic for Ω, we obtain the following equations :

$$\begin{cases} aJB + \sqrt{1-a^2}\begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} = \sqrt{1-a^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ a(\det B - 1).J + \sqrt{1-a^2}({}^tB\begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}B) = -aJ \end{cases}$$

These equations are equivalent to :

$$B = \frac{\sqrt{1-a^2}}{a}\begin{pmatrix} 0 & \frac{1}{\lambda} - 1 \\ 1-\lambda & 0 \end{pmatrix};$$

therefore, we have in the new coordinates :

$$M' = \begin{pmatrix} \mathbf{1} & \frac{\sqrt{1-a^2}}{a}\begin{pmatrix} 0 & \frac{1}{\lambda} - 1 \\ 1-\lambda & 0 \end{pmatrix} \\ \mathbf{0} & \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \end{pmatrix}.$$

The end of the lemma is clear.

■

In order to do new symplectic perturbations, we prove :

Lemma 16 *The matrix :*

$$P = \begin{pmatrix} \alpha\mathbf{1} & \beta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ -\beta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \alpha\mathbf{1} \end{pmatrix}.$$

is symplectic for Ω *if and only if* $\alpha^2+\beta^2=1$. *We have :* $P(\mathcal{P}_1)=\mathcal{P}_1$ *and* $P(\mathcal{P}_2)=\mathcal{P}_2$. *Moreover, if* $\alpha=\sqrt{1-a^2}$ *and* $\beta=-a$, *the matrix* $\bar{M}=PM'$ *is hyperbolic with complex eigenvalues.*

Proof of lemma 16 : It is easy to verify that P is symplectic and that $\mathcal{P}_1$ and $\mathcal{P}_2$ are invariant by $P\bar{M}$. Let us compute :

$$\tilde{M}=P.M'=\begin{pmatrix} \alpha\mathbf{1} & \alpha\frac{\sqrt{1-a^2}}{a}\begin{pmatrix}0 & \frac{1}{\lambda}-1\\ 1-\lambda & 0\end{pmatrix}+\beta\begin{pmatrix}0 & \frac{1}{\lambda}\\ \lambda & 0\end{pmatrix}\\ -\beta\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} & -\beta\frac{\sqrt{1-a^2}}{a}\begin{pmatrix}1-\lambda & 0\\ 0 & \frac{1}{\lambda}-1\end{pmatrix}+\alpha\begin{pmatrix}\lambda & 0\\ 0 & \frac{1}{\lambda}\end{pmatrix}\end{pmatrix}.$$

The planes $\mathcal{P}_1$ and $\mathcal{P}_2$ are invariant by $\tilde{M}$. Thus to find the eigenvalues and eigenspaces of $\tilde{M}$, we only have to study the restrictions of $\tilde{M}$ to $\mathcal{P}_1$ and $\mathcal{P}_2$. Moreover, these planes are lagrangian (for Ω). Therefore, if the eigenvalues of $\tilde{M}_{|\mathcal{P}_1}$ are μ_1 and μ_2, the eigenvalues of $\tilde{M}_{|\mathcal{P}_2}$ are $\frac{1}{\mu_1}$ and $\frac{1}{\mu_2}$, and thus we know all the eigenvalues of $\tilde{M}$.
Let us study the matrix of $\tilde{M}_{|\mathcal{P}_1}$, which is :

$$M_1=\begin{pmatrix}\alpha & \alpha\frac{\sqrt{1-a^2}}{a}(\frac{1}{\lambda}-1)+\beta\frac{1}{\lambda}\\ -\beta & -\beta\frac{\sqrt{1-a^2}}{a}(\frac{1}{\lambda}-1)+\alpha\frac{1}{\lambda}\end{pmatrix};$$

the trace of M_1 is :

$$T=\alpha(1+\frac{1}{\lambda})-\beta\frac{\sqrt{1-a^2}}{a}(\frac{1}{\lambda}-1);$$

and the determinant is $\frac{\alpha^2+\beta^2}{\lambda}=\frac{1}{\lambda}$.
Let us assume that $\beta=-a$ and $\alpha=\sqrt{1-a^2}$. Then :

$$T=\sqrt{1-a^2}(1+\frac{1}{\lambda}+\frac{1}{\lambda}-1)=2\sqrt{1-a^2}\frac{1}{\lambda};$$

therefore, the discriminant is :

$$\Delta=\frac{T^2}{4}-\frac{1}{\lambda}=((1-a^2)-\lambda)\frac{1}{\lambda^2}<0,$$

as $\lambda>1$.
This implies that the two eigenvalues of M_1 are complex (and their modulus is different from 1 because the determinant of M_1 is $\frac{1}{\lambda}\in]0,1[$). Therefore the eigenvalues of M_2 are complex too (in fact they are inverse of the eigenvalues of M_2).

■

Now, using ideas similar to these contained in [2] and using our way to "multiply periodic points", we shall mix perturbations to obtain a completely elliptic periodic point.

We fix y as before, with primary period τ, and call (identifying matrix and endomorphism) :

$$\forall i \in \mathbb{N}^*, M_i = Df(f^{i-1}y).$$

Lemma 17 *For every $\varepsilon > 0$, there exists an ε-perturbation $(\tilde{M}_i)_{i\geq 1}$ of $(M_i)_{i\geq 1}$ and $N \geq 1$ such that :*

$$\begin{cases} (\tilde{M}_{N\tau} \dots \tilde{M}_1)\mathbb{R}e_2 = S; \\ (\tilde{M}_{N\tau} \dots \tilde{M}_1)S = \mathbb{R}e_2. \end{cases}$$

Proof of lemma 17 : We can perturb M_τ in $\hat{M}_\tau$ such that :

- $M_{\tau|U+S}$ doesn't change;
- $\hat{M}_\tau \dots M_{1|E}$ is a small rotation R such that there exists $n_1 \geq 1$ such that : $R^{n_1}e_2 = e_1$.

Then we have :

$$\begin{cases} (\hat{M}_\tau \dots M_1)^{n_1}\mathbb{R}e_2 = \mathbb{R}e_1; \\ (\hat{M}_\tau \dots M_1)^{n_1}S = S. \end{cases}$$

Now, using lemma 16, we may replace M_τ by $\bar{M}_\tau = PM_\tau$. Then $\bar{M}_\tau \dots M_1$ is hyperbolic with complex eigenvalues and $\mathcal{P}_1$ and $\mathcal{P}_2$ are invariant. Doing another small perturbation of $\bar{M}_\tau$ (named $\bar{M}_\tau$ too), we may assume that there exists $n_2 \geq 1$ such that :

$$(\bar{M}_\tau \dots M_1)^{n_2}\mathbb{R}e_1 = S.$$

Then, there exists $s_0 \in S$ such that :

$$(\bar{M}_\tau \dots M_1)^{n_2}S = \mathbb{R}(e_1 + s_0).$$

Then, if $n_3 \geq 1$, we have :

$$\begin{cases} (M_\tau \dots M_1)^{n_3}(\bar{M}_\tau \dots M_1)^{n_2}\mathbb{R}e_1 = S \\ (M_\tau \dots M_1)^{n_3}(\bar{M}_\tau \dots M_1)^{n_2}S = \mathbb{R}(e_1 + \frac{1}{\lambda^{n_3}}s_0) \end{cases}$$

If n_3 is big enough, $e_1 + \frac{1}{\lambda^{n_3}}s_0$ is very close to e_1 and we can perturb M_τ in M'_τ (which preserve S) to have :

$$(M'_\tau \dots M_1)\mathbb{R}(e_1 + \frac{1}{\lambda^{n_3-1}}s_0) = \mathbb{R}e_1.$$

Now we can perturb M_τ in M''_τ such that $M''_{\tau|U+S} = M_{\tau|U+S}$ and $(M''_\tau \dots M_1)_{|E}$ is a small rotation such that : $(M''_\tau \dots M_1)^{n_4}e_1 = e_2$; then we have : $(M''_\tau \dots M_1)^{n_4}S = S$. Finally, we have the lemma for :

- $N = n_1 + n_2 + n_3 + n_4$;

- $\forall j \in \{1, \ldots, n_1\}, \tilde{M}_{j\tau} = \hat{M}_\tau$;
- $\forall j \in \{n_1+1, \ldots, n_1+n_2\}, \tilde{M}_{j\tau} = \bar{M}_\tau$;
- $\forall j \in \{n_1+n_2+1, \ldots, n_1+n_2+n_3-1\}, \tilde{M}_{j\tau} = M_\tau$;
- $\tilde{M}_{(n_1+n_2+n_3)\tau} = M'_\tau$;
- $\forall j \in \{n_1+n_2+n_3+1, \ldots, n_1+n_2+n_3+n_4\}, \tilde{M}_{j\tau} = M''_\tau$;
- the other M_j are unchanged.

■

Lemma 18 *There exist $N \geq 1$ and an ε-perturbation $(M'_i)_{i\geq 1}$ of $(M_i)_{i\geq 1}$ such that :*

$$\begin{cases} (M'_{N\tau} \ldots M'_1)e_2 = \pm e_2 \\ (M'_{N\tau} \ldots M'_1)s_0 = \pm s_0) \end{cases}$$

Proof of lemma 18 : Using lemma 16, we can perturb M_τ in PM_τ such that $P.M_\tau \ldots M_1$ has four complex eigenvalues, and doing another small perturbation we may assume that these eigenvalues are roots of some reals.
Thus we obtain $\bar{M}_\tau$ near M_τ such that :

$$\begin{cases} (\bar{M}_\tau \ldots M_1)^{n_0} e_2 = \sqrt{\lambda}^{n_0} e_2 \\ (\bar{M}_\tau \ldots M_1)^{n_0} s_0 = \frac{1}{\sqrt{\lambda}^{n_0}} s_0) \end{cases}$$

Then, if we define :

- $\tilde{M} = \tilde{M}_{N\tau} \ldots \tilde{M}_1$;
- α and β are such that : $\tilde{M}e_2 = \alpha s_0$ end $\tilde{M}s_0 = \beta e_1$;

then we have :

$$\begin{cases} (*) & (\bar{M}_\tau \ldots M_1)^{n_0 n} \tilde{M} (\bar{M}_\tau \ldots M_1)^{n_0 m} \tilde{M} e_2 = \alpha\beta \sqrt{\lambda}^{n_0(n-m)} e_2; \\ & (\bar{M}_\tau \ldots M_1)^{n_0 n} \tilde{M} (\bar{M}_\tau \ldots M_1)^{n_0 m} \tilde{M} s_0 = \alpha\beta \sqrt{\lambda}^{n_0(m-n)} s_0. \end{cases}$$

If $n = m$ is big, $|\alpha\beta|^{\frac{1}{2n}}$ is close to 1 and we can do small perturbations of $\bar{M}_\tau$ to multiply the vectors of the orbit of e_2 and s_0 by $\frac{1}{|\alpha\beta|^{\frac{1}{2n}}}$ and in $(*)$ e_2 is sent on e_2 or $-e_2$, and s_0 is sent on s_0 or $-s_0$.

■

Now, we can notice that $M'_{N\tau} \ldots M'_1$ is completely elliptic because its restriction to the lagrangian plane $S + \mathbb{R}e_2$ is elliptic.

Now, to finish the proof of the theorem, we replace y by z having a period which is a multiple of $N\tau$ (we multiply again the periodic points) and the same M_i than y (by using Franks' lemma along the orbit of y). Then we know by the previous lemma that we can perturb the M_i in M'_i and use Franks' lemma to conclude.

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