Convergence of the semi-group of Lax-Oleinik: a geometric point of view

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Abstract

Let M be a compact manifold, and $L : TM \to \mathbb{R}$ be a C^2 superlinear and strictly convex Lagrangian. Let $\hat{T}_t : C^0(M,\mathbb{R}) \to C^0(M,\mathbb{R})$ be the Lax-Oleinik semi-group defined by

$$\hat{T}_t u(x) = \inf_{\gamma}\{u(\gamma(0)) + \int L(\gamma(s),\dot{\gamma}(s))ds + c_0 t\}$$

where the infinum is taken among the continuous and C^1-piecewise arcs $\gamma : [0,t] \to M$ such that $\gamma(t) = x$. For one value of c_0, this semi-group has fixed points and a result of A. Fathi asserts that for all $u \in C^0(M,\mathbb{R})$, $(\hat{T}_t u)$ converges uniformly to one of these fixed points, u_-. We prove that the family of the adherences of the graphs of $(dT_t u)$ converges for the topology of Hausdorff to the adherence of the graph of du_-.

1 Introduction

Let M be a compact manifold endowed with a Riemannian metric. We will note (x, v) a point of the tangent bundle TM with $x \in M$ and v a vector tangent at x. The projection $\pi : TM \to M$ is then $(x, v) \to x$. The notation (x, p) will design a point of the cotangent bundle T^*M with $p \in T^*_x M$. and $\pi^* : T^*M \to M$ will be the canonical projection $(x, p) \to x$.

We consider a Lagrangian $L : TM \to \mathbb{R}$ which is C^2 and :

- uniformly superlinear : uniformly on $x \in M$, we have : $\lim\limits_{\|v\|\to+\infty} \dfrac{L(x,v)}{\|v\|} = +\infty$;

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- strictly convex : for all $(x,v) \in TM$, $\frac{\partial^2 L}{\partial v^2}(x,v)$ is positive definite.

We can associate to such a Lagrangian a C^2 Hamiltonian $H \ : T^*M \to \mathbb{R}$ and the Legendre transformation $\mathcal{L} \ : TM \to T^*M$ defined by : $\mathcal{L}(x,v) = \frac{\partial L}{\partial v}(x,v)$ which is a fibered C^1 diffeomorphism. The Hamiltonian H is then superlinear, strictly convex in the fiber and C^2. We note (f_t) the Euler-Lagrange flow associated to L and (ϕ_t) the Hamiltonian flow associated to H; then : $\phi_t = \mathcal{L} \circ f_t \circ \mathcal{L}^{-1}$.

The semi-group of Lax-Oleinik is the semi-group of non-linear operators $(T_t) \ : C^0(M,\mathbb{R}) \to C^0(M,\mathbb{R})$ defined by :

$$T_t u(x) = \inf_{\gamma} \{u(\gamma(0)) + \int L(\gamma(s), \dot{\gamma}(s)) ds\}$$

where the infinum is taken among the continuous and C^1-piecewise arcs $\gamma \ : [0,t] \to M$ such that $\gamma(t) = x$.

When u is semi-concave, a more geometric vision of this semi-group is given in [B2] (see also [B1] in dimension 1 and [F3] when u is C^2) : the action de (T_t) on u is seen as the action of the Hamiltonian flow (ϕ_t) on the graph $\mathcal{G}_{du}$ of du (at the points where du is defined) and the "suppression" of certain parts of the set $\phi_t(\mathcal{G}_{du})$ to obtain an exact (but not necessarly continuous) graph, the graph of $dT_t u$ (at the points it is defined). We will precise at the begining of the section 2 the precise statements which exist concerning this interpretation.

In [F1], A. Fathi proves that there exists a unique $c_0 \in \mathbb{R}$ such that the semi-group $\hat{T}_t \ : u \to T_t u + c_0 t$, $t \geq 0$ has a fixed point, and he proves in [F2] that for every $u \in C^0(M,\mathbb{R})$, the uniform limit $\lim_{t\to+\infty} \hat{T}_t u$ exists and is a fixed point of $(\hat{T}_t)$.

For $t > 0$, every $T_t u$ is Lipschitz and so allmost everywhere differentiable. Motivated by the geometric interpretation of the Lax-Oleinik group, we can ask ourselves if the convergence is not better than only uniform, and if we don't have the convergence (in a sense we will soon precise) of the graphs $\mathcal{G}_{dT_t u}$ to the graph $\mathcal{G}_{du_-}$ where u_- is a fixed point of $(\hat{T}_t)$ (we don't write the hat because for the derivative, the constant $c_0 t$ doesn't matter). The answer is positive in a sense we will now explained.

Definition. Let (X,d) be a metric space. Let $\mathcal{K}(X)$ be the set of non empty compact subsets of X. The Haudorff metric d_H is defined by :

$$\forall K_1, K_2 \in \mathcal{K}(X), d_H(K_1,K_2) = \max\{\sup_{x\in K_1} d(x,K_2), \sup_{x\in K_2} d(x,K_1)\}.$$

We also define : $\rho(K_1,K_1) = \sup_{x\in K_1} d(x,K_2)$.
Then : $d_H(K_1,K_2) = \max\{\rho(K_1,K_2), \rho(K_2,K_1)\}$.

NOTATIONS. Let $u \in C^0(M,\mathbb{R})$ and $t > 0$. Then $T_t u$ is allmost everywhere differentiable. Let $\mathrm{dom}(dT_t u)$ be the domain of definition of $dT_t u$. Then $\mathcal{G}(dT_t u)$ is the graph (in T^*M) of this function and $\bar{\mathcal{G}}(dT_t u)$ is its adherence.

Theorem 1 *Let $u \in C^0(M,\mathbb{R})$ and $u_- = \lim\limits_{t\to+\infty} \hat{T}_t u$. Then :*

$$\lim_{t\to+\infty} d_H(\bar{\mathcal{G}}(dT_t u), \bar{\mathcal{G}}(du_-)) = 0.$$

Hence, in the sense of the Hausdorff metric, the graphs of $dT_t u$ tend to the graph of du_- when t tends to $+\infty$.

Let us compare our result with the result obtained in [B1] by P. Bernard in the case of the forced Burgers equation on the circle; in his article, a convergence for the Hausdorff metric is proved, but the sets considered are not the same than ours : when the solution y of the Burger is C^1, these sets are the $C(\bar{\mathcal{G}}(dT_t u))$, where if $A \subset TM$, $C(A) = \bigcup_{x\in M} \mathrm{Conv}(A \cap T_x M)$ and Conv design the convex hull in the fiber. Thus he obtains the convergence of the fiberwise convexified graphs $C(\bar{\mathcal{G}}(dT_t u))$.

Via the geometric interpretation of the semi-group of Lax-Oleinik, we deduce two corollaries :

Corollary 2 *Let $u \in C^0(M,\mathbb{R})$, $u_- = \lim\limits_{t\to+\infty} \hat{T}_t u$ and $t_0 > 0$. There exists a decreasing family $(E_t)_{t\geq t_0}$ of subsets of M such that if we define $G_t = \overline{\phi_{t-t_0}(\mathcal{G}(dT_{t_0} u_{|E_t}))}$, then*

$$\lim_{t\to+\infty} d_H(G_t, \bar{\mathcal{G}}(du_-)) = 0.$$

Corollary 3 *Let $u : M \to \mathbb{R}$ be a C^1 function and $u_- = \lim\limits_{t\to+\infty} \hat{T}_t u$. There exists a decreasing family $(K_t)_{t\geq 0}$ of compact subsets of M such that $\phi_t(\bar{\mathcal{G}}(du_{|K_t}))$ converges for the Hausdorff distance to $\bar{\mathcal{G}}(du_-)$ when t tends to $+\infty$.*

Remark We may ask ourselves what happens when t tends to $+\infty$. Then $K_\infty = \bigcap K_t$ is a non empty compact subset of M such that for every $x \in K_\infty$, $(\pi^* \circ \Phi_t(x, du(x)))_{t\geq 0}$ is minimizing and therefore is in the stable manifold of the Aubry set. Then we find again by another way a result proved by P. Bernard in [B2] (he proves his result when u is semi-concave) : $\mathcal{G}(du) \cap W^s(\mathcal{A}^*) \neq \emptyset$ where $W^s(\mathcal{A}^*)$ is the stable manifold of the Aubry set $\mathcal{A}^*$ in T^*M.

2 Proof of the theorem 1

For $u \in C^0(M, \mathbb{R})$ and $t > 0$, we will note :

$$\Gamma_{x,t}(u) = \{\gamma \in C^1([0,t], M); \gamma(t) = x \quad \text{and} \quad \hat{T}_t u(x) = u(\gamma(0)) + \int_0^t L(\gamma, \dot{\gamma}) ds + c_0 t\}.$$

Let us notice that every element of $\Gamma_{x,t}(u)$ is a minimizer of the Lagrangian action with fixed ends. It is known that this set is not empty (see e.g. [F3] and [B2]), and we have :

Proposition 4 *Let $u \in C^0(M, \mathbb{R})$, $t > 0$ and $x \in M$. Then :*

1. *$T_t u$ is differentiable at x if and only if $\Gamma_{x,t}(u) = \{\gamma_x\}$ is one point; in this case : $dT_t u(x) = \frac{\partial L}{\partial v}(x, \dot{\gamma}_x(t))$; therefore $(x, dT_t u(x)) = \phi_t \circ \mathcal{L}(\gamma_x(0), \dot{\gamma}_x(0))$;*
2. *for every $s \in]0, t[$ and every $\gamma \in \Gamma_{x,t}(u)$, $T_s u$ is differentiable at $\gamma(s)$ and $dT_s u(\gamma(s)) = \frac{\partial L}{\partial v}(\gamma(s), \dot{\gamma}(s))$; therefore $(x, dT_s u(x)) = \phi_s \circ \mathcal{L}(\gamma_x(0), \dot{\gamma}_x(0))$;*
3. *if u is semi-concave, we have :*

$$\forall t > 0, \bar{\mathcal{G}}(dT_t u) \subset \phi_t(\mathcal{G}(du)).$$

The proofs of this proposition are contained in [F3] and [B2]. The definition of a semi-concave function is given in [F3] and [B2]; for example, a C^1 function is always semi-concave. Another result contained in [F3] is the lemma of “à priori compactness”, which implies :

Lemma 5 *For every $t > 0$, there exists $k_t > 0$ such that : $\forall x \in M, \forall u \in C^0(M, \mathbb{R}), \forall \gamma \in \Gamma_{x,t}(u), \forall s \in [0, t], \|\dot{\gamma}(s)\| \leq k_t$.*

We note then : $B(k) = \{(x, v) \in TM; \|v\| \leq k\}$ and define for every $t > 0$: $U_t : C^0(M, \mathbb{R}) \times B(k_t) \to \mathbf{R}$ by : $U_t(u, (x, v)) = u(\gamma(-t)) + \int_{-t}^0 L(\gamma_v(s), \dot{\gamma}_v(s)) ds + c_0 t$ where γ_v is the solution of the Euler-Lagrange equations satisfying : $\gamma_v(0) = v$. Using the lemma 5, we have :

$$\forall t > 0, \forall u \in C^0(M, \mathbb{R}), \forall x \in M, \hat{T}_t u(x) = \min\{U_t(u, (x, v)); v \in B(k_t)\}.$$

Moreover, if for every $u \in C^0(M, \mathbb{R})$ and $x \in M$ we note :
$M_{x,t}(u) = \{v \in B(k_t); U_t(u, (x, v)) = \min\{U_t(u, (x, w)); w \in T_x M\}\}$, then : $\Gamma_{x,t}(u) = \{\gamma_v(. - t); v \in M_{x,t}(u)\}$.

With these new notations, we can reformulate the 1 of the proposition 4 :
$\mathcal{G}(dT_t u) = \mathcal{L}(\bigcup_{x \in M, \sharp M_{x,t}(u) = 1} M_{x,t}(u))$. The gain of this presentation is that now the considered set of arcs is compact. We can then use the following topological lemma :

Lemma 6 *Let (Y,d) a metric space and $pr : Z \to Y$ a continuous bundle whose fiber is a compact metric space. Let $f : Z \to \mathbb{R}$ be a continuous map. One define $F : Y \to \mathcal{K}(Z)$ by : $F(y) = \{x \in pr^{-1}(y); f(x) = \min\limits_{t \in pr^{-1}(y)} f(t)\}$. Then F is upper semi-continuous, that is :*

$$\forall \varepsilon > 0, \forall y_0 \in Y, \exists \alpha > 0, \forall y \in Y, d(y, y_0) < \alpha \Rightarrow \rho(F(y), F(y_0)) < \varepsilon.$$

In particular, the graph of F is closed and at every point $y \in Y$ where $\sharp F(y) = 1$, F is continuous.

Proof Let us fix $\varepsilon > 0$ and $y_0 \in Y$. We choose one $x_0 \in F(y_0)$ and define $W = \{z \in Z; d(z, F(y_0)) < \varepsilon\}$; then, W is an open neighbourhood of $F(y_0)$, and by the definition of F, the continuity of f and the fact that the fiber is compact :

$$\min_{x \in pr^{-1}(y_0) \backslash W} f(x) = f(x_0) + \varepsilon_0 > f(x_0).$$

We then define :

$$U = \{y \in Y; \exists z \in pr^{-1}(y), f(z) < f(x_0) + \frac{\varepsilon_0}{3} \quad \text{and} \quad \forall x \in pr^{-1}(y) \backslash W; f(x) > f(x_0) + \frac{2\varepsilon_0}{3}\}.$$

Let us prove that U is a neighbourhood of y_0. If it is not true, there exists a sequence (y_n) in Y which converges to y_0 such that : $\forall n \in \mathbb{N}^*, \exists x_n \in pr^{-1}(y_0) \backslash W; f(x_n) \leq f(x_0) + \frac{2\varepsilon_0}{3}$. Extracting a subsequence and using the fact that the fiber is compact, we can assume that (x_n) converges to $x_\infty \in pr^{-1}(y_0) \backslash W$. Then x_∞ is such that : $f(x_\infty) \leq f(x_0) + \frac{2\varepsilon_0}{3}$, and this contradicts the definition of ε_0. Thus U is a neighbourhood of y_0.

Let us consider $y \in U$. Then necessarily $m(y) = \min\{f(x); x \in pr^{-1}(y)\} < f(x_0) + \frac{\varepsilon_0}{3}$ and thus $F(y) \backslash W = \emptyset$ so $F(y) \subset W$ i.e. $\rho(F(y), F(y_0)) < \varepsilon$. Thus we have found a neighbourhood U of y_0 such that : $\forall y \in U; \rho(F(y), F(y_0)) < \varepsilon$.

The end of the lemma is just a general property of semi-continuous functions. ■

We then apply the lemma 6 to $Y = C^0(M, \mathbb{R}) \times M$, $Z = C^0(M, \mathbb{R}) \times B(k_t)$ and $f = U_t$ (then $F(u, x) = M_{x,t}(u)$) for a $t > 0$ and obtain :

1. $\forall \varepsilon > 0, \forall u_0 \in C^0(M, \mathbb{R}), \forall x_0 \in M, \exists \alpha > 0, \forall u \in C^0(M, \mathbb{R}), \forall x \in M$, $(d(x, x_0) < \alpha$ and $\|u - u_0\|_\infty < \alpha) \Rightarrow \rho(M_{x,t}(u), M_{x_0,t}(u_0)) < \varepsilon$;

2. $\bigcup\limits_{x \in M} M_{x,t_0}(u_0)$ is closed (it is the graph of "F" intersected with $\{u = u_0\}$).

3. if u_0 is differentiable at x, then (u_0, x) is a point of continuity of "F" (we refind in particular a result contained in [F3] : du_0 is continuous on its domain).

Let us introduce a new set :

DEFINITION. For every $u \in C^0(M,\mathbb{R})$ and $t_0 > 0$, we define :

$$\tilde{\mathcal{G}}(dT_{t_0}u) = \mathcal{L}\left(\bigcup_{x\in M} M_{x,t_0}(u)\right).$$

Then we deduce from the lemma 6 :

Lemma 7 *Let $u_0 \in C^0(M,\mathbb{R})$, $t_0 > 0$ and $\varepsilon > 0$; there exists $\alpha > 0$ such that :*

$$\forall u \in C^0(M,\mathbb{R}), \|u - u_0\|_\infty < \alpha \Rightarrow \rho(\tilde{\mathcal{G}}(dT_{t_0}u), \tilde{\mathcal{G}}(dT_{t_0}u_0)) < \varepsilon.$$

PROOF We find for every $x \in M$ a α_x such that :

$$\forall u \in C^0(M,\mathbb{R}), \forall y \in M,$$

$$\|u_0 - u\|_\infty < \alpha_x \quad \text{and} \quad d(x,y) < \alpha_x \Rightarrow \rho(M_{y,t_0}(u), M_{x,t_0}(u_0)) < \varepsilon.$$

M being compact, we find a finite subset $\{x_1, \dots, x_N\}$ of M and $\alpha > 0$ such that for every $u \in C^0(M,\mathbb{R})$ satisfying $\|u - u_0\|_\infty < \alpha$ and every $x \in M$, there exists $i \in \{1, \dots, N\}$ such that $\rho(M_{x,t_0}(u), M_{x_i,t_0}(u_0)) < \varepsilon$. This implies that if $\|u - u_0\|_\infty < \alpha$, then

$$\rho(\bigcup_{x\in M} M_{x,t_0}(u), \bigcup_{x\in M} M_{x,t_0}(u_0)) \leq \varepsilon.$$

The Legendre transformation being continuous, changing eventually ε and thus α, we deduce that :

$$\|u - u_0\|_\infty < \alpha \Rightarrow \rho(\tilde{\mathcal{G}}(dT_{t_0}u), \tilde{\mathcal{G}}(dT_{t_0}u_0)) \leq \varepsilon.$$

■

We can precise the links between $\bar{\mathcal{G}}(dT_{t_0}u)$ and $\tilde{\mathcal{G}}(dT_{t_0}u)$:

Lemma 8 *Let $t_0 > 0$.*

1. *if $u \in C^0(M,\mathbb{R})$, $\bar{\mathcal{G}}(dT_{t_0}u) \subset \tilde{\mathcal{G}}(dT_{t_0}u)$;*
2. *if u_- is a fixed point of $(\hat{T}_t)$, then we have the equality : $\bar{\mathcal{G}}(du_-) = \tilde{\mathcal{G}}(du_-)$.*

Let us notice that the part 2. of the lemma is in fact contained in [F3].

PROOF 1. $\tilde{\mathcal{G}}(dT_{t_0}u)$ is a closed subset of T^*M containing $\mathcal{G}(dT_{t_0}u)$, therefore : $\bar{\mathcal{G}}(dT_{t_0}u) \subset \tilde{\mathcal{G}}(dT_{t_0}u)$.
2. Let u_- a fixed point of $(\hat{T}_t)$. We have to prove that :

$$\bigcup_{x\in M} M_{x,t_0}(u_-) \subset \overline{\bigcup_{x\in M, \sharp M_{x,t_0}(u_-)=1} M_{x,t_0}(u_-)}.$$

Let $x \in M$ and let $v \in M_{x,t_0}(u_-)$. Then by the 2. of the proposition 4, for every $s \in]0, t_0[$, $u_- = T_{t_0-s}u_-$ is differentiable at $x_s = \pi(f_{-s}(v))$ and $du_-(x_s) = \phi_{-s}(\mathcal{L}(v))$. Thus :

$$\forall s \in]0, t_0[, M_{x_s,t_0} = \{f_{-s}(x,v)\} \quad \text{and} \quad \lim_{s\to 0} f_{-s}(x,v) = (x,v);$$

which implies that $v \in \overline{\bigcup_{x\in M, \sharp M_{x,t_0}(u_-)=1} M_{x,t_0}(u_-)}$. ■

Let now $u \in C^0(M,\mathbb{R})$ and $u_- = \lim_{t\to+\infty} \hat{T}_t u$. Fix $\varepsilon > 0$. From lemmas 7 and 8, we deduce that there exists $\tau > 0$ such that :

$$\forall t \geq \tau, \rho(\bar{\mathcal{G}}(dT_t u), \bar{\mathcal{G}}(du_-)) = \rho(\bar{\mathcal{G}}(dT_t u), \tilde{\mathcal{G}}(du_-)) \leq \rho(\tilde{\mathcal{G}}(dT_t u), \tilde{\mathcal{G}}(du_-)) \leq \varepsilon.$$

We have then proved a sort of "semi-convergence" of $(\bar{\mathcal{G}}(dT_t u))$ to $\bar{\mathcal{G}}(du_-)$ when t tends to $+\infty$, and we have to prove now that for t big enough, $\rho(\bar{\mathcal{G}}(du_-), \bar{\mathcal{G}}(dT_t u)) \leq \varepsilon$. To obtain that, we prove :

Lemma 9 *Let $u_0 \in C^0(M,\mathbb{R})$, $t_0 > 0$ and $\varepsilon > 0$; there exists $\alpha > 0$ such that :*

$$\forall u \in C^0(M,\mathbb{R}), \|u - u_0\|_\infty < \alpha \Rightarrow \rho(\bar{\mathcal{G}}(dT_{t_0}u_0), \bar{\mathcal{G}}(dT_{t_0}u)) < \varepsilon$$

Then the theorem comes from this lemma : we fix $u \in C^0(M,\mathbb{R})$, $t_0 > 0$ and $\varepsilon > 0$. Let $u_0 = \lim_{t\to+\infty} \hat{T}_t u$. By lemma 9, we can associate to u_0, t_0 and ε a certain α. Let us choose $\tau > 0$ such that for every $t \geq \tau$, we have : $\|T_t u - u_0\|_\infty < \alpha$. Then for every $t > \tau + t_0$, we have : $\rho(\bar{\mathcal{G}}(u_0), \bar{\mathcal{G}}(dT_t u)) = \rho(\bar{\mathcal{G}}(dT_{t_0}u_0), \bar{\mathcal{G}}(dT_{t_0}(T_{(t-t_0)}u))) < \varepsilon$.

PROOF We know that $\mathcal{G}(dT_{t_0}u_0)$ and $\mathcal{G}(dT_{t_0}u)$ are graphs almost everywhere above the zero-section. Let us fix $\varepsilon > 0$. We can find a finite family $\{x_1, \dots, x_N\}$ in $\mathrm{dom}(dT_{t_0}u_0)$ such that :

$$\bar{\mathcal{G}}(dT_{t_0}u_0) \subset \bigcup_{n=1}^{N} B(dT_{t_0}u_0(x_n), \frac{\varepsilon}{2}).$$

Then we choose $\varepsilon_0 > 0$ such that : for all $(x,v), (y,w) \in B(k_t)$: $d((x,v),(y,w)) < \varepsilon_0 \Rightarrow d(\mathcal{L}(x,v), \mathcal{L}(y,w)) < \frac{\varepsilon}{2}$.

As $T_{t_0}u_0$ is differentiable at every x_i, we have : $\Gamma_{x_i,t_0}(u_0) = \{\gamma_i\}$ is one point and thus $\sharp M_{x_i,t_0}(u_0) = 1$. Thus (x_i, u_0) is a point of continuity of the function $F\ : (x, u) \to M_{x,t}(u)$. Thus there exists $\alpha_i > 0$ such that :

$$\forall u \in C^0(M, \mathbb{R}), \forall y \in M, \|u - u_0\|_\infty < \alpha_i \quad \text{and} \quad d(x_i, y) < \alpha_i$$

$$\Rightarrow d_H(M_{x_i,t_0}(u_0), M_{y,t_0}(u)) < \varepsilon_0 \Rightarrow d_H(\mathcal{L}(M_{x_i,t_0}(u_0)), \mathcal{L}(M_{y,t_0}(u))) < \frac{\varepsilon}{2}.$$

Let us now consider $u \in C^0(M, \mathbb{R})$ such that $\|u - u_0\| < \alpha = \min \alpha_i$. Then $T_{t_0}u$ is allmost everywhere differentiable. Thus for every $n \in \{1, \dots, N\}$, there exists $y_n \in B(x_i, \alpha)$ where $T_{t_0}u$ is differentiable. We deduce :

$$\forall n \in \{1, \dots, N\}, d(dT_{t_0}u(y_n), dT_{t_0}u_0(x_n)) = d_H(\mathcal{L}(M_{x_i,t_0}(u_0)), \mathcal{L}(M_{y_i,t_0}(u))) < \frac{\varepsilon}{2},$$

therefore

$$\bar{\mathcal{G}}(dT_{t_0}u_0) \subset \bigcup_{n=1}^{N} B(dT_{t_0}u_0(x_n), \frac{\varepsilon}{2}) \subset \bigcup_{n=1}^{N} B(dT_{t_0}u(y_n), \varepsilon);$$

and then $\rho(\bar{\mathcal{G}}(dT_{t_0}u_0), \bar{\mathcal{G}}(dT_{t_0}u)) < \varepsilon$. ■

Now the proof of the theorem 1 is complete.

3 Proof of the corollaries

To deduce the corollary 2 from the theorem, we remark, using the 2. of the proposition 4, that for every $t \geq t_0$, there exists $E_t \subset M$ such that $\mathcal{G}(dT_t u) = \phi_{t-t_0}(\mathcal{G}(dT_{t_0}u_{|E_t}))$. Moreover, (E_t) is decreasing because (T_t) is a semi-group. We take the adherence of this equality and use the theorem to conclude.

To prove the corollary 3, assuming that u is C^1 and then semi-concave, we use the 3. of the proposition 4 which implies that for every $t \geq 0$, there exists $E_t \subset M$ such that :

$$\bar{\mathcal{G}}(dT_t u) = \phi_t(\mathcal{G}(du_{|E_t})).$$

(T_t) being a semi-group, (E_t) is decreasing. The flow (Φ_t) being continuous, every $\mathcal{G}(du_{|E_t})$ is compact and thus every E_t is compact.

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