

# The 2-link periodic orbits which maximize or minimize the length of a $p$ -dimensional Birkhoff billiard are hyperbolic

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## Abstract

We prove that in a  $p$ -dimensional billiard, every 2-link non-degenerate periodic solution which minimizes or maximizes the length is hyperbolic. This result generalizes the results of D. Treshchev concerning the case of the planar Birkhoff billiard.

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# 1 Introduction.

It is well-known that the symplectic diffeomorphisms may have stably different types of periodic points (elliptic or hyperbolic), and that near these different sorts of periodic points completely different qualitative behaviours may occur. It is well-known too that a lot of symplectic diffeomorphisms have a “generating function” (by “a lot”, we mean an  $C^1$ -open neighbourhood of the identity map in the case of Weinstein generating function, or an  $C^1$ -open neighbourhood of completely integrable maps in the case of Lagrangian generating function), and that a critical point of the generating function is a fixed point of the initial diffeomorphism. If the critical point of the generating function is non-degenerate, we can associate with it a (Morse) index. An old question is then : is there a relation between this index and the type of the corresponding fixed point of the diffeomorphism (is it elliptic, hyperbolic...)? A lot of articles deal with this question :

- in [1], [2] and [3], I proved that certain extrema of (Weinstein) generating functions correspond in general to completely elliptic fixed points of symplectic diffeomorphisms;
- in [10], D. Offin proves that in certain symmetric cases, the minimum of the Lagrangian action is hyperbolic; a theorem of R. Mañé asserts that every “strong minimum” (i.e. contained in the so-called critical level; it is a very strong notion) is hyperbolic (see [9] for the theorem and [5] for the proof). I have proved in [4] that the minimum of the Lagrangian action is not always hyperbolic (by using examples constructed in [1]).

In the case of billiards, there exists a very natural “Lagrangian generating function” : the length  $L$ . This function  $L$  is useful to find periodic orbits with different bounces (see for example [7]). Moreover, in [7], a precise classification of the type of periodic points is given in the case of planar billiards; for example, it is shown that each non-degenerate maximum of the length is hyperbolic, and that each minimum with an even number of bounces is hyperbolic too. The method uses some calculus made in [8] (for area preserving diffeomorphisms). In higher dimension, there is no obvious link between the index of a critical point of the length and the type of the corresponding periodic point. In fact, in [6], the calculus made in [8] for one-degree-of-freedom systems is given in every dimension, but as it is noticed by the authors, this calculus is not sufficient to give the type of the considered periodic orbits (it only gives the Lefschetz index of the periodic point).

We will in this article give a generalization of the result concerning the extrema of the length in every dimension but for 2-link orbits. We don’t know if the result is true for every even period.

Let us describe the setting :  $M$  is a  $C^2$ -hypersurface of  $\mathbf{R}^{p+1}$  (with  $p \geq 1$ ). Let  $\mathcal{M} = T^1\mathbf{R}_{|M}^{p+1}$  be the unitary tangent bundle restricted to  $M$  : it is a  $2p$ -dimensional  $C^1$ -submanifold of  $\mathbf{R}^{2p+2}$ . If  $(x, v) \in \mathcal{M}$ , we say :

- If  $M \cap \{x+tv; t > 0\} = \emptyset$ , the billiard map is not defined at  $(x, v)$ ; if  $M \cap \{x+tv; t > 0\} \neq \emptyset$  and if for  $t_0 = \inf\{t > 0; x+tv \in M\}$ ,  $x+t_0v \notin M$  or  $M$  and  $\{x+tv; t \geq 0\}$  are tangent at  $x+t_0v$ , we say that the billiard map is not defined at  $(x, v)$ .
- At the other points of  $\mathcal{M}$  (and the set of these points is open in  $\mathcal{M}$ ), we define  $t_0 = \inf\{t > 0; x+tv \in M\}$ . Then  $x+t_0v \in M$ . Let  $H$  be the tangent hyperplane to  $M$  at  $x+t_0v$ ; then  $B(x, v) = (x+t_0v, s(v))$  where  $s$  is the reflection across  $H$ .

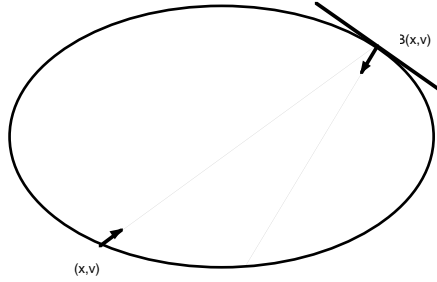


Figure 1

Let us notice that our definition is slightly different of the usual one : the billiard map  $B$  may be non-defined at the vectors tangent to  $M$ . This fact is not important because we will work “far” from these vectors.

It is well-known that the non-trivial periodic orbits of  $B$  correspond to critical points of the length function. More precisely, for the case of the 2-periodic orbits, we have :

- we define  $L : M \times M \rightarrow \mathbf{R}$  by :  $L(x, y) = \|x - y\|$  where  $\|\cdot\|$  is the usual Euclidian norm.
- then  $(x, v)$  is a (non trivial) 2-periodic point of  $B$  if and only if, if we write  $(y, w) = B(x, v)$ ,  $(x, y)$  is a critical point of  $L$ . Then we have of course :  $v = \frac{y-x}{\|y-x\|}$  is a vector normal to  $M$  at  $x$  and at  $y$ .

We will say that  $x$  (resp.  $y$ ) is a non-degenerate critical point of a function  $f$  if its Hessian  $\text{Hess}f(x)$  is non-degenerate.

**Theorem 1** *Let  $(x, y)$  be a non-degenerate maximizer or minimizer of the length function defined on  $M \times M$ ; then  $(x, \frac{y-x}{\|y-x\|})$  is a hyperbolic 2-periodic point of the billiard map.*

As every strictly convex billiard has a maximum for the length and as generically, the critical points of the length are non-degenerate (see [7]) , we deduce :

**Corollary 2** *Let  $M$  be a strictly convex billiard generic for the  $C^2$  topology. Then it has a 2-periodic hyperbolic orbit.*

## 2 Proof of the theorem.

Instead of working in the coordinates  $(x, v) \in \mathcal{M}$ , we will use the coordinates  $(x, x') \in M \times M$  where  $v = \frac{x'-x}{\|x-x'\|}$  (it is well-known that these are local coordinates, see for example [7]). The billiard map in these coordinates is  $\mathcal{B}$ , and if  $(x, y)$  is a critical point of  $L$ , we have easily that, if  $D\mathcal{B}(x, y)(u, v) = (v, w)$  and  $D\mathcal{B}(y, x)(v, w) = (w, t)$ , then (the calculus is given in [7] and [6]) :

- $L_{x,x'}(x, y)u + 2L_{x',x'}(x, y)v + L_{x,x'}(x, y)w = 0;$
- $L_{x',x}(x, y)v + 2L_{x,x}(x, y)w + L_{x',x}(x, y)t = 0.$

If  $a$  and  $b$  are orthogonal to  $x - y$ , an easy computation gives :  $L_{x,x'}(x, y)(u, v) = -\frac{\langle a, b \rangle}{\|x-y\|}$ . Therefore  $L_{x,x'}(x, y)$  is invertible and we have :

$$\begin{aligned} D\mathcal{B}^2(x, y) \begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} 0 & \mathbf{1} \\ -L_{x',x}^{-1}L_{x',x} & -2L_{x',x}^{-1}L_{x,x} \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1} \\ -L_{x,x'}^{-1}L_{x,x'} & -2L_{x,x'}^{-1}L_{x',x'} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\ &= \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & -2L_{x',x}^{-1}L_{x,x} \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & -2L_{x,x'}^{-1}L_{x',x'} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\ &= \begin{pmatrix} -\mathbf{1} & -2L_{x,x'}^{-1}L_{x',x'} \\ 2L_{x',x}^{-1}L_{x,x} & -\mathbf{1} + 4L_{x',x}^{-1}L_{x,x}L_{x,x'}^{-1}L_{x',x'} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \end{aligned}$$

Moreover, we know that the following symmetric matrix is definite :

$$\text{Hess}L(x, y) = \begin{pmatrix} L_{x,x} & L_{x',x} \\ L_{x,x'} & L_{x',x'} \end{pmatrix}$$

Let us prove that, with this hypothesis, the matrix  $D\mathcal{B}^2(x, y)$  has no eigenvalue with modulus 1.

Let us assume that  $\lambda$  is an eigenvalue of  $D\mathcal{B}^2(x, y)$ , and let  $(u, v)$  be an eigenvector for  $\lambda$ . We have :

$$\begin{aligned} \lambda u &= -u - 2L_{x,x'}^{-1}L_{x',x'}v; \\ \lambda v &= 2L_{x',x}^{-1}L_{x,x}u + (-\mathbf{1} + 4L_{x',x}^{-1}L_{x,x}L_{x,x'}^{-1}L_{x',x'})v. \end{aligned}$$

If  $\lambda = -1$ , as  $L_{x,x}$  and  $L_{x',x'}$  are definite, we find  $u = v = 0$  and thus  $-1$  is not an eigenvalue. If  $\lambda \neq -1$ , we find :

$$\begin{aligned} u &= -\frac{2}{\lambda+1}L_{x,x'}^{-1}L_{x',x'}v; \\ 0 &= \left(4\lambda L_{x',x}^{-1}L_{x,x}L_{x,x'}^{-1}L_{x',x'} - (\lambda+1)^2\right)v. \end{aligned}$$

Thus, if  $\mu = e^{i\theta}$  is a square root of  $\lambda$ , we only have to find the kernel of :

$$L_{x',x}^{-1}L_{x,x}L_{x,x'}^{-1}L_{x',x'} - \frac{1}{4}\left(\mu + \frac{1}{\mu}\right)^2\mathbf{1} = L_{x',x}^{-1}L_{x,x}L_{x,x'}^{-1}L_{x',x'} - \cos^2\theta\mathbf{1}$$

But as  $\text{Hess}L(x, y)$  is definite, the matrix :

$$\begin{pmatrix} \mu & 0 \\ 0 & \bar{\mu} \end{pmatrix} \begin{pmatrix} L_{x,x} & L_{x',x} \\ L_{x,x'} & L_{x',x'} \end{pmatrix} \begin{pmatrix} \bar{\mu} & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} L_{x,x} & \lambda L_{x',x} \\ \bar{\lambda} L_{x,x'} & L_{x',x'} \end{pmatrix}$$

is definite too with the same sign. Therefore the sum of the two matrices is definite :

$$M_\lambda = \begin{pmatrix} 2L_{x,x} & (1+\lambda)L_{x',x} \\ (1+\bar{\lambda})L_{x,x'} & 2L_{x',x'} \end{pmatrix}$$

Let us write that a vector  $(z_1, z_2)$  is in its kernel :

$$\begin{aligned} 2L_{x,x}z_1 + (1+\lambda)L_{x',x}z_2 &= 0 \\ (1+\bar{\lambda})L_{x,x'}z_1 + 2L_{x',x'}z_2 &= 0 \end{aligned}$$

These equations are equivalent to :

$$\begin{aligned} z_1 &= -\frac{2}{1+\bar{\lambda}}L_{x,x'}^{-1}L_{x',x}z_2; \\ 0 &= \left(L_{x',x}^{-1}L_{x,x}L_{x,x'}^{-1}L_{x',x'} - \frac{1}{4}(\lambda+1)(\bar{\lambda}+1)\right)z_2. \end{aligned}$$

As  $\mu = e^{i\theta}$  is a square root of  $\lambda$ , we have :  $\frac{1}{4}(\lambda+1)(\bar{\lambda}+1) = \cos^2\theta$  and thus we know in fact that :

$$\ker \left( L_{x',x}^{-1}L_{x,x}L_{x,x'}^{-1}L_{x',x'} - \cos^2\theta\mathbf{1} \right) = \{0\}.$$

Finally, we have proved that  $D^2\mathcal{B}(x, y)$  has no eigenvalue with modulus 1 and therefore  $(x, y)$  is a hyperbolic periodic point.

## Bibliography.

- [1] M.-C. Arnaud. Type des points fixes des difféomorphismes symplectiques de  $\mathbf{T}^n \times \mathbf{R}^n$ , *Supplément au Bull. Soc. Math. Fr.* **48** (1992).
- [2] M.-C. Arnaud. Type of critical points of Hamiltonian functions and of fixed points of symplectic diffeomorphisms, *Nonlinearity* **7** (1994) 1281-1290.
- [3] M.-C. Arnaud. Existence d'orbites périodiques complètement elliptiques sur des hypersurfaces convexes présentant certaines symétries, *C.R. Acad. Sci. Paris série I* **328** (1999) 1035-1038.
- [4] M.-C. Arnaud. On the type of certain periodic orbits minimizing the Lagrangian action, *Nonlinearity* **11** (1998) 143-150.
- [5] G. Contreras & R. Iturriaga. Convex Hamiltonians without conjugate points, *Ergod. Th. & Dynam. Sys.* **19** (1999) 901-952.
- [6] H. Kook & J. Meiss. Periodic orbits for symplectic maps, *Physica D* **35** (1989) 65-86.
- [7] V. Kozlov & D. Treshchev. Billiards : a genetic introduction in the dynamics of systems with impacts, *Transl. of math. Monograph. of A.M.S.* **89** (1991).
- [8] R. MacKay & J. Meiss. Linear stability of periodic orbits in Lagrangian systems, *Physics Letters* **98 A** (1983), 92-94.
- [9] R. Mañé. Lagrangian flows : the dynamics of globally minimizing orbits, *Bol. Soc. Bras. Mat.* **28** (1997) 141-153.
- [10] D. Offin. A spectral theorem for reversible second order equations with periodic coefficients, *Differential and integral equations* **5** (1992) 615-629.