Are pseudographs Lagrangian submanifolds?

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Abstract

Let $H\ : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian defined on the cotangent bundle of a compact and connected manifold and $u\ : M \to \mathbb{R}$ be a semi-concave function. If $\mathcal{E}(u)$ is the set of all super-differentials of u and (φ_t) the Hamiltonian flow of H, we prove that for $t > 0$ small enough, $\varphi_{-t}(\mathcal{E}(u))$ is an exact Lagrangian Lipschitz graph; we deduce a geometric proof of a result due to Fathi-Siconolfi and Bernard : such a Hamiltonian has always $C^{1,1}$ subsolutions.
Moreover, using the Lax-Oleinik semi-group $(\breve{T}_t)$, we prove that for $t > 0$ small enough, $\varphi_{-t}(\mathcal{E}(u))$ is the graph of $d\breve{T}_t u$. Hence the Lipschitz pseudographs that P. Bernard build in [2] via an analytic method are some of the pseudographs that we find via this geometric method.

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1 Introduction

In the recent developments of the so-called “weak K.A.M. theory”, the notion of “pseudograph” did appear recently in an article of P. Bernard (see [1]) to prove some results concerning Arnold’s and Mather’s diffusion. Let us explain quickly how.
We consider the Hamilton-Jacobi equation : $H(x, du(x)) = C$ for a Hamiltonian function $H : T^*M \to \mathbb{R}$ defined on a cotangent bundle which is C^2, superlinear and convex in the fiber. In the 1980’s, P.-L. Lions and M. Crandall introduced the notion of viscosity solution for this equation (see [5]). In the case where $M = \mathbb{T}^n$, Lions, Papanicolaou and Varadhan proved the existence of a viscosity solution. Then, in [6], A. Fathi proved the existence of a viscosity solution (which he called a weak K.A.M. solution) for any manifold. Such a weak K.A.M. solution is semi-concave and hence locally Lipschitz (see for example [7]). A semi-concave function $u : M \to \mathbb{R}$ is Lipschitz and hence differentiable on a set $E \subset M$ with full (Lebesgue) measure, and the graph $\mathcal{G}(u) = \{(q, du(q)); q \in E\}$ of the derivatives of any semi-concave function is what we call a *pseudograph.*

When u is C^2, the pseudograph $\mathcal{G}(u)$ is in fact a graph above the whole manifold M and is a Lagrangian graph.

That’s why a very natural question is :

Questions : are the pseudographs Lagrangian manifolds in general? And, as a pseudograph is not a smooth manifold, in which sense?

Let us notice that in the other sense, M. Chaperon proved in [4] that every Lagrangian submanifold of T^*M which is Hamiltonianly isotopic to the zero section can be “cut” in such a way that we obtain the graph of the differential of a Lipschitz function defined on M. In certain cases, A. Ottolenghi & C. Viterbo proved in [9] that this Lipschitz function is a semi-concave, and hence the “cut graph” is in fact a pseudograph.

Before explaining which kind of positive answer we give to the questions, let us introduce some notions. At first, we recall what is a semi-concave function and we define the enlarged pseudographs.

DEFINITION.

1. Let U be an open subset of $\mathbb{R}^d$, $K \geq 0$ be a constant and $u : U \to \mathbb{R}$ be a function. We say that u is K-semi-concave if for every $x \in U$, there exists a linear form p_x defined on $\mathbb{R}^d$ such that :

$$\forall y \in U, u(y) \leq u(x) + p_x(y - x) + K\|y - x\|^2;$$

(where $\|.\|$ is the usual Eucidian norm). Then we say that p_x is a K-super-differential of u at x.

2. Let M be a compact and connected manifold with a finite atlas $\mathcal{A} = \{(U_i, \Phi_i : U_i \to \mathbb{R}^d); 1 \leq i \leq N\}$ and $u : M \to \mathbb{R}$ be a function; we say that u is K-semi-concave if for every $i \in \{1, \dots, N\}$, the function $u \circ \Phi_i^{-1} : \Phi_i(U_i) \to \mathbb{R}$ is K-semi-concave. Then, a K-superdifferential of u is a $p_x \circ D\Phi_i(x)$ where p_x is a K-super-differential of $u \circ \Phi_i^{-1}$ at $\Phi_i(x)$.
3. A function is semi-concave if it is K-semi-concave for a certain K; while the quantitative notion of "K-semi-concave function" depends on the considered atlas of M that we choose, the notion of "semi-concave function" is independent of this atlas. The notion of super-differential too doesn't depend on the atlas.
4. if $u : M \to \mathbb{R}$ is semi-concave, its *enlarged pseudograph* is the set $\mathcal{E}(u)$ of all the super-differentials of u :

$$\mathcal{E}(u) = \{(x, p_x); p_x \quad \text{is a superdifferential of} \quad u \quad \text{at} \quad x\}.$$

The enlarged pseudograph $\mathcal{E}(u)$ of a semi-concave function u contains its pseudograph $\mathcal{G}(u)$; in general, $\mathcal{E}(u)$ is no longer a graph and $\mathcal{E}(u)$ is compact (it's clearly closed and P. Bernard proved in [1] that it is bounded).

REMARK. In fact, even if it doesn't appear in the notation, the definition of $\mathcal{E}(u)$ depends on the choice of the constant K of semi-concavity that we choose, and in the proofs we will fix such a constant K. But a posteriori, because of proposition 4, we see that $\mathcal{E}(u)$ is independant of this constant.

For a survey of the principal properties of the semi-concave functions, the reader may have a look at the appendix of [1] and the book [7].

Let us now explain which kind of manifold will interest us :

DEFINITION. Let M be a d-dimensional compact and connected manifold.

1. a non-empty subset N of T^*M is a d-dimensional *Lipschitz submanifold* of T^*M if for every $x \in N$, there exists a (smooth) chart (U, Φ) of T^*M at x such that $\Phi(N \cap U)$ is the graph of a Lipschitz map $\ell : V \to \mathbb{R}^d$ defined on a open subset V of $\mathbb{R}^d$. Of course, this notion is invariant by C^1-diffeomorphism.
2. a Lipschitz graph is $\{s(x); x \in M\}$ where $s : M \to T^*M$ a Lipschitz section. Of course, a Lipschitz graph is a d-dimensional Lipschitz submanifold of T^*M.
3. a d-dimensional Lipschitz submanifold N of T^*M is *exact Lagrangian* if it is exact Lagrangian in the sense of distributions, that is if for every $\gamma : [a, b] \to N$

closed Lipschitz arc drawn on N, we have : $0 = \int_\gamma \lambda$ (where λ designates the Liouville 1-form of T^*M). This notion is invariant under C^1 exact symplectic diffeomorphisms.

Then the Lipschitz graph of a Lipschitz section $s : M \to T^*M$ is exact Lagrangian if and only if there exists a $C^{1,1}$ function $u : M \to \mathbb{R}$ (that is a C^1 function whose derivative is Lipschitz) such that : $s = du$.

The result that we obtain is :

Theorem 1 *Let M be a compact and connected manifold and $u : M \to \mathbb{R}$ be a semi-concave function. Then its enlarged pseudograph $\mathcal{E}(u)$ is an exact Lagrangian Lipschitz submanifold of M.*

To prove this result, we will use any Hamiltonian function $H : T^*M \to \mathbb{R}$ which is C^2, superlinear and convex in the fiber (such a Hamiltonian is called a Tonelli Hamiltonian); denoting by (φ_t) its Hamiltonian flow, we will prove that for $t > 0$ enough small, $\varphi_{-t}(\mathcal{E}(u))$ is an exact Lagrangian Lipschitz graph.
A corollary of this proof is a new (and more geometric) proof of a result of Fathi-Siconolfi-Bernard (see [8] and [2]) :

Corollary 2 *Any Tonelli Hamiltonian has a $C^{1,1}$ subsolution.*

The precise definition of subsolution will be given later.

Let us compare our solution with the one given by P. Bernard in [2] : using the Lax-Oleinik semi-groups $(T_t)_{t>0}$ and $(\breve{T}_t)_{t>0}$ associated to the Tonelli hamiltonian H, he proves :

Proposition (P. Bernard) *For each semi-concave function $u : M \to \mathbb{R}$, for every $t > 0$ small enough, the function $\breve{T}_t u$ is $C^{1,1}$.*

Let us notice that :

Proposition 3 *Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian and $u : M \to \mathbb{R}$ be a semi-concave function; then there exists $\varepsilon > 0$ such that for every $t \in]0, \varepsilon]$, $\breve{T}_t u$ is $C^{1,1}$ and $\mathcal{G}(\breve{T}_t u) = \varphi_{-t}(\mathcal{E}(u))$.*

Hence the analytic construction of P. Bernard corresponds exactly to our geometric construction.

2 Proof of theorem 1

As we explained before, we will prove :

Proposition 4 *Let M be a d-dimensional compact and connected manifold, let $H : T^*M \to \mathbb{R}$ be any Tonelli Hamiltonian function and let $u : M \to \mathbb{R}$ be any semi-concave function. Then there exists $\varepsilon > 0$ such that, for every $t \in [-\varepsilon, 0[$, $\varphi_t(\mathcal{E}(u))$ is an exact Lagrangian Lipschitz graph.*

Of course, the theorem is a consequence of this proposition. Let us now prove this proposition.

We assume that M is a compact and connected manifold with a finite atlas $\mathcal{A}$ and $u : M \to \mathbb{R}$ is a K-semi-concave function. We consider any Tonelli Hamiltonian function $H : T^*M \to \mathbb{R}$ and denote by $(\varphi_t)_{t\in\mathbb{R}}$ its Hamiltonian flow.

2.1 Proof that $\varphi_t(\mathcal{E}(u))$ is a graph for $t \in [-\varepsilon, 0[$

Given $\varepsilon \in]0, 1]$ small enough, we want to know if it is possible that for a $t \in [-\varepsilon, 0[$, $\varphi_t(\mathcal{E}(u))$ is not a graph above a certain part of M.
If $x = (q, p) \in T^*M$, we denote by $\mathcal{V}(x)$ its vertical : $\mathcal{V}(x) = \{y \in T^*M; \pi(y) = \pi(x) = q\}$ where $\pi : T^*M \to M$ designates the usual projection.
Then we want to know if it is possible for a $t \in]0, \varepsilon]$ and a $x \in \mathcal{E}(u)$ that $\mathcal{V}(\varphi_{-t}(x)) \cap \varphi_{-t}(\mathcal{E}(u))$ contains at least two points. It means that there exists two different points $(q_0, p_0), (q_1, p_1) \in \mathcal{E}(u)$ such that $(q_1, p_1) \in \varphi_t(\mathcal{V}(\varphi_{-t}(q_0, p_0)))$.
We write ; $\forall \tau \in [-\varepsilon, 0], (q_j(\tau), p_j(\tau)) = \varphi_\tau(q_j, p_j)$. Then :
$(q_1(-t), p_1(-t)) \in \mathcal{V}(\varphi_{-t}(q_0, p_0)) = \mathcal{V}(q_0(-t), p_1(-t))$.
If ε is chosen small enough, because $\mathcal{E}(u)$ is compact, such a thing may happen only for q_0, q_1 and $(\pi \circ \varphi_\tau(q_0(-t), sp_0(-t) + (1 - s)p_1(-t)))_{\tau\in[0,\varepsilon],s\in[0,1]}$, in a same chart of the atlas $\mathcal{A}$. Then from now we work in the coordinates given by such a chart, i.e. in $\mathbb{R}^d$ and $T^*\mathbb{R}^d = \mathbb{R}^d \times \mathbb{R}^d$ and we write : $(q_1, p_1) = \varphi_t(q_0(-t), p_0(-t) + \Delta p)$ with $t \in]0, \varepsilon]$.
Because H is strictly convex in the fiber, there exists two constants $C > c > 0$ such that :

$$\forall \tau \in [-1, 1], \forall (q, p) \in \mathcal{E}(u), \forall v \in \mathbb{R}^d, c\|v\|^2 \leq H_{p,p}(\varphi_\tau(q, p))(v, v) \leq C\|v\|^2.$$

Denoting the second derivative in the variables p by $H_{p,p}$, we have :

$$\varphi_t(q_1(-t), p_1(-t)) - \varphi_t(q_0(-t), p_0(-t)) = \int_0^1 D\varphi_t(q_0(-t), p_0(-t) + s\Delta p)(0, \Delta p)ds.$$

Using the linearized Hamilton equations, we obtain that the quantity in the integral is equal to :

$$(t(H_{p,p}(\varphi_t(q_0(-t),p_0(-t)+s\Delta p)\Delta p+\|\Delta p\|\varepsilon_1(s,t,\Delta p)),\Delta p+\|\Delta p\|\varepsilon_2(s,t,\Delta p))$$

where the functions ϵ_j tend uniformly to 0 when t tends to 0.
We deduce :

$$(p_1-p_0)(q_1-q_0)=t\int_0^1\left[H_{p,p}(\varphi_t((q_0(-t),p_0(-t)+s\Delta p)(\Delta p,\Delta p)+t\|\Delta p\|^2\eta(s,t,\Delta p)\right]ds$$

where the function η tends uniformly to 0 when t tends to 0.
We deduce that if ε has been chosen small enough :

$$(p_1-p_0)(q_1-q_0)\geq\frac{c}{2}t\|\Delta p\|^2\quad\text{and}\quad\|q_1-q_0\|\leq 2Ct\|\Delta p\|.$$

Moreover, we know too that p_j is a K-super-differential of u at q_j, for $j=0,1$. Hence :
$u(q_1)-u(q_0)\leq p_0(q_1-q_0)+K\|q_1-q_0\|^2$;
$u(q_0)-u(q_1)\leq p_1(q_0-q_1)+K\|q_1-q_0\|^2$.
We deduce by adding up these two inequalities :

$$(p_1-p_0)(q_1-q_0)\leq 2K\|q_1-q_0\|^2\leq 2CKt^2\|\Delta p\|^2.$$

Finally, we have proved that there exists two strictly positive constants c and C such that :

$$\frac{c}{2}t\|\Delta p\|^2\leq(p_1-p_0)(q_1-q_0)\leq 2CKt^2\|\Delta p\|^2.$$

It is obviously impossible for $t>0$ small enough and $\Delta p\neq 0$.

2.2 Proof that $\pi\circ\varphi_t(\mathcal{E}(u))=M$

We want to prove that for $t\in[-\varepsilon,0[$, the graph $\varphi_t(\mathcal{E}(u))$ covers the whole M.
We have found in subsection 2.1 a constant ε such that :

$$\forall t\in]0,\varepsilon],\forall x\in T^*M,\sharp(\varphi_t(\mathcal{V}(x))\cap\mathcal{E}(u))\leq 1$$

where we denote by $\sharp(E)$ the number of elements of E. Now we want to prove that :

$$\forall t\in]0,\varepsilon],\forall x\in T^*M,\varphi_t(\mathcal{V}(x))\cap\mathcal{E}(u)\neq\emptyset.$$

As in subsection 2.1, we will work in the charts of a finite atlas of M.

Let us denote by $\bar{B}_v(q,R)$ the closed ball in the vertical $\mathcal{V}(q,0)$ centered at $(q,0)$ with radius $R\geq 0$. Then we prove :

Lemma 5 *For every $R > 0$, there exists $\varepsilon_0 \in]0, \varepsilon[$ such that, for every $q \in M$ and every $t \in [-\varepsilon_0, \varepsilon_0]\backslash\{0\}, \varphi_t(\bar{B}_v(q, R))$ is the graph of $dg_{t,q}$ where $g_{t,q} : K_{t,q} \to \mathbb{R}$ is a C^2 function.*

PROOF Given any $q \in M$, we know that $\varphi_t(\bar{B}_v(q, R))$ is a Lagrangian submanifold (with boundary) because the vertical is a Lagrangian submanifold and φ_t is symplectic. If moreover $\varphi_t(\bar{B}_v(q, R))$ is the graph of a C^1 function above a part $K_{t,q}$ of M, because a ball is connected and homologically trivial, there exists a (unique) C^2 function $g_{t,q} : K_{t,q} \to \mathbb{R}$ such that $g_{t,q}(q) = 0$ and $\varphi_t(\bar{B}_v(q, R))$ is the graph of $dg_{t,q}$.

If $R > 0$ is fixed, there exists $\varepsilon_0 \in]0, \varepsilon[$ such that, for every $q \in M$ and every $t \in [-\varepsilon_0, \varepsilon_0]\backslash\{0\}, \varphi_t(\bar{B}_v(q, R))$ is the graph of a C^1 function above a part $K_{t,q}$ of M. Indeed, in a similar way to the proof in the previous subsection :

$$\varphi_t(q, p + \Delta p) - \varphi_t(q, p) = \int_0^1 D\varphi_t(q, p + s\Delta p)(0, \Delta p)ds$$

and then this quantity is equal to :

$$\int_0^1 (t(H_{p,p}(\varphi_t(q, p + s\Delta p))\Delta p + \|\Delta p\|\varepsilon_1(s, t, \Delta p)), \Delta p + \|\Delta p\|\varepsilon_2(s, t, \Delta p))ds$$

where the functions ϵ_j tend uniformly (for $p, p + \Delta p \in \bar{B}_v(q, R), q \in M,\ s \in [0, 1]$) to 0 when t tends to 0. Hence if ε_0 is small enough and $t \in [-\varepsilon_0, \varepsilon_0]\backslash\{0\}$, we have for $p, p + \Delta p \in \bar{B}_v(q, R), q \in M$: $(\varphi_t(q, p + \Delta p) - \varphi_t(q, p)).(\Delta p, 0)$ has the same sign as $t \int_0^1 H_{p,p}(\varphi_t(q, p + s\Delta p)(\Delta p, \Delta p)ds$ which is strictly positive for $t > 0$, strictly negative for $t < 0$. Then $\pi \circ \varphi_t(q, p+\Delta p) \neq \pi \circ \varphi_t(q, p)$ and we obtain a graph. As every vertical is a C^∞ manifold and the flow is C^1, this graph is a C^1 manifold. As $D(\pi \circ \varphi_t)(q, p)(0, \delta p)$ is close to $tH_{p,p}(q, p)\delta p \neq 0$, this graph is transverse to the vertical at every point and then it's the graph of a C^1 function. $\square$

We want to use this function $g_{t,q}$ to prove that for $t \in [-\varepsilon, 0[$, the graph $\varphi_t(\mathcal{E}(u))$ covers the whole M. A problem is that when t tends to 0, the domain on which $g_{t,q}$ is defined tends to a point (when R is fixed), and it would be usefull for us to define $g_{t,q}$ on a domain with fixed size. That's why we modify our Hamiltonian :

Lemma 6 *Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian and $\mathcal{K} \subset T^*M$ be a compact subset; then there exists a Tonelli Hamiltonian function $H_0 : T^*M \to \mathbb{R}$ such that :*

- *$\forall x \in \mathcal{K}, H(x) = H_0(x)$;*
- *outside a compact subset of T^*M, H_0 is positively homogeneous with degree 2 in the fiber.*

We don't prove this lemma, easy to prove by using some bump functions.
Let us recall that we are interested in the $\varphi_t(\mathcal{E}(u))$ for $t \in [-\varepsilon, 0[$. The set $\mathcal{E}(u)$ being compact, we introduce the real : $K_0 = \max\{H(x); x \in \mathcal{E}(u)\}$. Then, $\mathcal{K} = H^{-1}(]-\infty, K_0])$ is compact and we find by lemma 6 a Tonelli Hamiltonian function $H_0 : T^*M \to \mathbb{R}$ such that :

- $\forall x \in \mathcal{K}, H(x) = H_0(x)$;
- outside a compact subset of T^*M, H_0 is positively homogeneous with degree 2 in the fiber.

Let us denote by (ψ_t) the Hamiltonian flow of H_0. As $H_{|\mathcal{K}} = H_{0|\mathcal{K}}$ and the Hamiltonian is constant along the trajectories, we have : $(\varphi_{t|\mathcal{K}})_{t\in\mathbb{R}} = (\psi_{t|\mathcal{K}})_{t\in\mathbb{R}}$. Hence : $\forall t \in \mathbb{R}, \psi_t(\mathcal{E}(u)) = \varphi_t(\mathcal{E}(u))$ and we have now to prove proposition 4 for H_0 instead of H.

Hence we assume that outside a compact $\mathcal{K}$ invariant by (φ_t), H is positively homogeneous in the fiber with degree 2. We prove :

Lemma 7 *Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian which is positively homogeneous with degree 2 in the fiber outside an invariant compact subset $\mathcal{K} \subset M_{\frac{R}{2}} = \bigcup_{q\in M} \bar{B}_v(q, \frac{R}{2})$ of T^*M. Then there exists four strictly positive constants c, C, ε_0 and r such that, for every $q \in M$ and every $t \in [-\varepsilon_0, \varepsilon_0]\backslash\{0\}$, the set $\varphi_t(\bar{B}_v(q_0, \frac{R\varepsilon_0}{|t|}))$ contains the graph of $dg_{t,q}$ where $g_{t,q} : \bar{D}(q) = \bar{B}(q, r) \to \mathbb{R}$ is a C^2 function such that :*

$$\forall t \in]0, \varepsilon_0], \forall q \in M, \forall Q \in \bar{D}(q), \frac{1}{4tC}\|v\|^2 \le D^2 g_{t,q}(Q)(v, v) \le \frac{4}{ct}\|v\|^2$$

and :

$$\forall t \in [-\varepsilon_0, 0[, \forall q \in M, \forall Q \in \bar{D}(q), \frac{4}{ct}\|v\|^2 \le D^2 g_{t,q}(Q)(v, v) \le \frac{1}{4Ct}\|v\|^2.$$

Proof Because H is positively homogeneous with degree 2 in the fiber outside an invariant compact subset, $H_{p,p}$ is uniformly bounded on T^*M, i.e. there exists two constants $C > c > 0$ such that :

$$\forall x \in T^*M, c\|v\|^2 \le H_{p,p}(x)(v, v) \le C\|v\|^2.$$

For every $\lambda \in \mathbb{R}_+^*$, let us denote by $\mathcal{H}_\lambda$ the map : $\mathcal{H}_\lambda : T^*M \to T^*M$ such that : $\mathcal{H}_\lambda(q, p) = (q, \lambda p)$. Then :

$$\forall (q, p) \notin \mathcal{K}, \mathcal{H}_\lambda(q, p) \notin \mathcal{K} \Rightarrow \forall t \in \mathbb{R}, \varphi_t \circ \mathcal{H}_\lambda(q, p) = \mathcal{H}_\lambda \circ \varphi_{t\lambda}(q, p).$$

We deduce :

$$\forall (q,p) \notin \mathcal{K}, \mathcal{H}_\lambda(q,p) \notin \mathcal{K} \Rightarrow \forall t \in \mathbb{R}, \forall (0,\delta p), D\varphi_t(q,\lambda p)(0,\delta p) = D\varphi_{t\lambda}(q,p)(0,\delta p).$$

Using lemma 5, we can associate to R an $\varepsilon_0 > 0$.
Moreover, we know through the linearized Hamilton equations that :
$D(\pi \circ \varphi_t)(q,p)(0,\delta p) = t(H_{p,p}(q,p) + \eta(q,p,t))\delta p$ where η is a function which tends to 0 when t tends to 0 uniformly for $(q,p) \in M_R$.
Then, if ε_0 is (uniformly in M_R) chosen small enough, we have :

- $\forall (q,p) \in M_R; \forall t \in [0,\varepsilon_0], \frac{ct}{2}\|\delta p\|^2 \leq (D(\pi \circ \varphi_t)(q,p)(0,\delta p)\delta p).\delta p \leq 2Ct\|\delta p\|^2$;
- $\forall (q,p) \in M_R; \forall t \in [-\varepsilon_0,0], \frac{ct}{2}\|\delta p\|^2 \geq (D(\pi \circ \varphi_t)(q,p)(0,\delta p)\delta p).\delta p \geq 2Ct\|\delta p\|^2$.

Let us now consider $(q,p) \in T^*M \backslash M_R$ and $t \in [-\frac{R\varepsilon_0}{\|p\|}, \frac{R\varepsilon_0}{\|p\|}]$. Then if we write : $p_0 = \frac{R}{\|p\|}p$, we have $\pi \circ \varphi_t(q,p) = \pi \circ \varphi_{t\frac{\|p\|}{R}}(q,p_0)$ is in the chart at q because $(q,p_0) \in M_R$ and $t\frac{\|p\|}{R} \in [-\varepsilon_0, \varepsilon_0]$. Hence we have :

- if $t \in]0, \frac{R\varepsilon_0}{\|p\|}]$, $(D(\pi \circ \varphi_t)(q,p)(0,\delta p)).\delta p = (D\varphi_{\frac{t\|p\|}{R}}(q,p_0)(0,\delta p)).\delta p \geq \frac{ct}{2}\|\delta p\|^2$;
- if $t \in [-\frac{R\varepsilon_0}{\|p\|}, 0[$, $(D(\pi \circ \varphi_t)(q,p)(0,\delta p)).\delta p = (D\varphi_{\frac{t\|p\|}{R}}(q,p_0)(0,\delta p)).\delta p \leq \frac{ct}{2}\|\delta p\|^2$.

Now let us fix $t \in [-\varepsilon_0, \varepsilon_0] \backslash \{0\}$, $q \in M$ and let us consider $\varphi_t(\bar{B}_v(q, \frac{R\varepsilon_0}{|t|}))$. Then $\pi(\varphi_t(\bar{B}_v(q, \frac{R\varepsilon_0}{|t|})))$ is in the same chart as q and :

- if $t \in]0, \varepsilon_0]$: $\forall (q,p) \in \bar{B}_v(q, \frac{R\varepsilon_0}{|t|}), (D(\pi \circ \varphi_t)(q,p)(0,\delta p)).\delta p \geq \frac{ct}{2}\|\delta p\|^2$;
- if $t \in [-\varepsilon_0, 0[$: $\forall (q,p) \in \bar{B}_v(q, \frac{R\varepsilon_0}{|t|}), (D(\pi \circ \varphi_t)(q,p)(0,\delta p)).\delta p \leq \frac{ct}{2}\|\delta p\|^2$.

On another side, we have :
$\forall (q,p_1), (q,p_2) \in \bar{B}_v(q, \frac{R\varepsilon_0}{|t|}), (\pi \circ \varphi_t(q,p_2) - \pi \circ \varphi_t(q,p_1)).(p_2 - p_1) =$

$$\left(\int_0^1 D(\pi \circ \varphi_t)(q, p_1 + s(p_2 - p_1))(p_2 - p_1) ds \right) (p_2 - p_1) ds.$$

We deduce :

- if $t \in]0, \varepsilon_0]$: $\forall (q,p_1), (q,p_2) \in \bar{B}_v(q, \frac{R\varepsilon_0}{|t|}), (\pi \circ \varphi_t(q,p_2) - \pi \circ \varphi_t(q,p_1)).(p_2 - p_1) \geq \frac{ct}{2}\|p_2 - p_1\|^2$;
- if $t \in [-\varepsilon_0, 0[$: $\forall (q,p_1), (q,p_2) \in \bar{B}_v(q, \frac{R\varepsilon_0}{|t|}), (\pi \circ \varphi_t(q,p_2) - \pi \circ \varphi_t(q,p_1)).(p_2 - p_1) \leq \frac{ct}{2}\|p_2 - p_1\|^2$.

Finally, $\pi \circ \varphi_t$ is injective in $\bar{B}_v(q, \frac{R\varepsilon_0}{|t|})$.

Moreover, we have seen that :
$\forall (q,p) \notin \mathcal{K}, \mathcal{H}_\lambda(q,p) \notin \mathcal{K} \Rightarrow \forall t \in \mathbb{R}, \forall (0,\delta p), D\varphi_t(q,\lambda p)(0,\delta p) = D\varphi_{t\lambda}(q,p)(0,\delta p)$.

Hence the tangent space to the Lagrangian submanifold $\varphi_t(\bar{B}_v(q, \frac{R\varepsilon_0}{|t|})$ is everywhere transverse to the vertical. Finally, $\varphi_t(\bar{B}_v(q, \frac{R\varepsilon_0}{|t|})$ is the graph of a C^1 function above a part of M, hence the graph of the differential of a C^2 function that we call again : $g_{t,q} : K_{t,q} \to \mathbb{R}$ such that $g_{t,q}(q) = 0$.

Moreover, $K_{t,q}$ is a homeomorphic to a d-dimensional closed ball and contains the (topological) sphere $S(q) = \{\varphi_{\varepsilon_0}(q,p); \|p\| = R\}$ for $t > 0$ and $S'(q) = \{\varphi_{-\varepsilon_0}(q,p); \|p\| = R\}$ for $t < 0$; then it contains the connected component $D(q)$ of $M \backslash S(q)$ (resp. $M \backslash S'(q)$) which is contained in the chart. Let us notice that this domain is independant of t and depends continuously on q. Hence there exists $r > 0$ such that :
for every $q \in M$, for every $t \in [-\varepsilon_0, \varepsilon_0] \backslash \{0\}$, $g_{t,q}$ is defined on the closed ball of M centered at q with radius r; we denote this ball again by $D(q)$ and from now we will consider $g_{t,q} : D(q) \to \mathbb{R}$.

Then for every $q \in M$ and every $Q \in D(q)$, $D^2 g_{t,q}(Q)$ is a symmetric matrix (in coordinates) whose graph is the image of the "linear vertical" $\ker D\pi(q,p)$ by $D\varphi_t(q,p)$ (for a $p \in \bar{B}_v(q, \frac{R\varepsilon_0}{|t|})$). We have proved before that for $t > 0$: $\forall (q,p) \in B_v(q, \frac{R\varepsilon_0}{|t|})$, $\frac{ct}{2}\|\delta p\|^2 \leq (D(\pi \circ \varphi_t)(q,p)(0, \delta p)).\delta p \leq 2Ct\|\delta p\|^2$. Moreover, in a similar way, we can deduce from the Hamilton linearized equations that in $M_{\frac{R\varepsilon_0}{|t|}}$, $D(\pi_p \circ \varphi_t)$ is (uniformly) very close to the identity. Then :

$$\forall t \in]0, \varepsilon_0], \forall q \in M, \forall Q \in \bar{D}(q), \frac{1}{4tC}\|v\|^2 \leq D^2 g_{t,q}(Q)(v,v) \leq \frac{4}{ct}\|v\|^2$$

and :

$$\forall t \in [-\varepsilon_0, 0[, \forall q \in M, \forall Q \in \bar{D}(q), \frac{4}{ct}\|v\|^2 \leq D^2 g_{t,q}(Q)(v,v) \leq \frac{1}{4Ct}\|v\|^2.$$

□

Now we choose a eventually smaller $\varepsilon_0 > 0$ such that : $K \leq \frac{1}{4C\varepsilon_0}$ where K is the constant of semi-concavity of u. Then we have (in coordinates) for $t \in]0, \varepsilon_0]$:
$g_{t,q}(Q + h) - g_{t,q}(Q) \geq dg_{t,q}(Q)h + \frac{1}{4Ct}\|h\|^2$;
$u(Q + h) - u(Q) \leq ph + K\|h\|^2$ where p is a K-super-differential of u at Q;
hence :
$(g_{t,q} - u)(Q + h) - (g_{t,q} - u)(Q) \geq (dg_{t,q}(Q) - p)h + \left(\frac{1}{4Ct} - K\right)\|h\|^2 \geq (dg_{t,q}(Q) - p)h$;
having a support linear form at every point, $g_{t,q} - u$ is convex, and even strictly convex if we have asked : $K < \frac{1}{4C\varepsilon_0}$; in this case $g_{t,q} - u$ has at most one minimizer and hence there exists at most one $Q \in D(q)$ such that 0 is a super-differential of $u - g_{t,q}$ at Q, i.e. such that $dg_{t,q}(Q)$ is a super-differential of u at Q, i.e. such that $dg_{t,q}(Q) \in \mathcal{E}(u)$. If such a Q exists, then $(Q, dg_{t,q}(Q)) \in \mathcal{E}(u) \cap \varphi_t(\mathcal{V}(q, 0))$, i.e. is the point that we want to find to conclude the proof of this subsection.

Hence we only have to prove that $g_{t,q} - u$ has a minimum in the interior $D(q)$ of $\bar{D}(q)$. Let us notice that :

- on M, u is uniformly bouded : $\|u\|_\infty \leq C_1$;
- there exists a constant C_2 such that : for $t \in [-\varepsilon_0, \varepsilon_0]$ and $q \in M$, $\|dg_{t,q}(q)\| = \|\Pi_p \circ \varphi_t(\tilde{q})\| \leq C_2$;
- for every $Q \in D(q)$ and $t \in]0, \varepsilon_0]$, we have : $g_{t,q}(Q) = \int_0^1 dg_{t,q}(q + s(Q-q))(Q-q)ds = \int_0^1 \left(dg_{t,q}(q)(Q-q) + s\int_0^1 D^2 g_{t,q}(q + s\tau(Q-q))(Q-q, Q-q)d\tau \right) ds$ and then $g_{t,q}(Q) \geq -C_2\|Q-q\| + \frac{1}{8tC}\|Q-q\|^2$. Taking $t > 0$ small enough (uniformly), we obtain that $g_{t,q}$ is very big on the boundary $\partial D(q)$.

As we know that $g_{t,q}(q) = 0$, we deduce that the minimum of $g_{t,q} - u$ is attained in the interior of $D(q)$ and finish the proof.

2.3 Proof that $\varphi_t(\mathcal{E}(u))$ is a Lipschitz graph

We have proved that for $t \in]0, \varepsilon_0]$, $\varphi_{-t}(\mathcal{E}(u))$ is a graph above M. Because this graph is compact ($\mathcal{E}(u)$ is compact), it's the graph of a continuous section $s_t : M \to T^*M$. We have to prove that s_t is Lipschitz. We may eventually change ε_0 in such a way that $K < \frac{1}{C\varepsilon_0}$.
We will use the so-called Bouligand's paratingent cone :

Definition. Let E be a subset of T^*M. The paratingent cone to E at $(q,p) \in E$ is defined (in chart but it doesn't depend on the chart) as the subset of $T_{(q,p)}(T^*M)$ whose elements are the limits of the sequences :
$\left(\frac{1}{t_n}(q_n - q_n'), \frac{1}{t_n}(p_n - p_n') \right)_{n\in\mathbb{N}}$ with $t_n \in \mathbb{R}_+^*$, $q_n, q_n', p_n, p_n' \in E$ and $\lim q_n = \lim q_n' = q$, $\lim p_n = \lim p_n' = p$. It is denoted by $C_{(q,p)}E$.

If $(q,p), (q',p') \in \mathcal{E}(u)$ are in a same chart, we have seen that : $(p'-p)(q'-q) \leq 2K\|q'-q\|^2$. We deduce that for all $(\delta q, \delta p) \in C_{(q,p)}\mathcal{E}(u)$, we have : $\delta p.\delta q \leq 2K\|\delta q\|^2$. Moreover, we know that for every $(q,p) \in \mathcal{E}(u)$, there exists $q_0 \in M$ such that $q \in D(q_0)$ and $p = dg_{t,q_0}(q)$. We have seen too that : $D^2 g_{t,q_0}(q)(v,v) \geq \frac{4}{ct}\|v\|^2$. Hence, because $2K < \frac{4}{c\varepsilon_0} < \frac{4}{ct}$, if $\mathcal{T}(q,p)$ is the set of the lines contained in the tangent space to the graph of dg_{t,q_0} at (q,p), we have : $\mathcal{T}(q,p) \cap C_{(q,p)}\mathcal{E}(u) = \emptyset$.

If $V(q,p) = \ker D\pi(q,p)$ designates the linear vertical and $\mathcal{P}_V(q,p)$ the set of the lines contained in $V(q,p)$, we deduce from the previous inequality that : $\mathcal{P}_V(q,p) \cap C_{(q,p)}\left(\varphi_{-t}(\mathcal{E}(u))\right) = \emptyset$.

Finally, we have proved that the paratingent cone to $\varphi_{-t}(\mathcal{E}(u))$, which is the graph of s_t, contains no vertical line. Let us deduce that s_t is Lipschitz. We assume that there are two sequences of points (q_n, p_n), (q_n', p_n') of $\varphi_{-t}(\mathcal{E}(u))$ such that $\lim \frac{\|p_n' - p_n\|}{d(q_n', q_n)} = +\infty$. Using a subsequence, because $\varphi_{-t}(\mathcal{E}(u))$ is compact, we may assume that the two sequences converge. Then necessarily (q_n) and (q'_n) have the same limit (because the previous limit is $+\infty$ and $\|p'_n - p_n\|$ is bounded). Hence by continuity of s_t, (p_n) and (p'_n) too have the same limit. But if we write $t_n = \|p_n - p'_n\|$ and if we use a subsequence in such a way that $(\frac{p'_n - p_n}{\|p'_n - p_n\|})$ converges to a u, we obtain that $\lim_{n\to\infty} \frac{1}{t_n}(q'_n - q_n, p'_n - p_n) = (0, u)$ is in the paratingent cone to $\varphi_{-t}(\mathcal{E}(u))$ at (q, p), it contradicts the fact that this paratingent cone contains no vertical line. Hence s_t is Lipschitz.

2.4 Proof that $\varphi_t(\mathcal{E}(u))$ is an exact Lagrangian Lipschitz graph

We have to prove that there exists a C^1 function (hence it will be $C^{1,1}$) $u_t : M \to \mathbb{R}$ such that $s_t = du_t$. It is enough to prove that for any closed Lipschitz arc $\gamma : [a, b] \to M$, then $\int_a^b s_t(\gamma(\tau))\dot\gamma(\tau)d\tau = 0$. Let us define a closed loop of T^*M by : $\forall \tau \in [a, b], \eta(\tau) = (\eta_1(\tau), \eta_2(\tau)) = \varphi_t(\gamma(\tau), s_t(\gamma(\tau)))$. The arc γ being Lipschitz and s_t being Lipschitz, the arc η is Lipschitz too. Hence we can define $\int_\eta \lambda$ where λ is the Liouville 1-form. The flow being exact symplectic, we have : $\int_\eta \lambda = \int_a^b s_t(\gamma(\tau))\dot\gamma(\tau)d\tau$. We are reduced to compute $\int_\eta \lambda = \int_a^b \eta_2(\tau)\dot\eta_1(\tau)d\tau$.

Let us recall that η is a closed Lipschitz arc drawn on $\mathcal{E}(u)$; then $\eta_2(\tau)$ is a K-super-differential of u at $\eta_1(\tau)$ and :

$$u(\eta_1(\tau + \delta\tau)) - u(\eta_1(\tau)) \le \eta_2(\tau)(\eta_1(\tau + \delta\tau) - \eta_1(\tau)) + K\|\eta_1(\tau + \delta\tau) - \eta_1(\tau)\|^2.$$

Moreover, u, η_1, η_2 and hence $u \circ \eta_1$ are Lipschitz, then (Lebesgue) almost everywhere derivable. If τ is a point where $u \circ \eta_1$ and η_1 are derivable, we obtain by dividing by δt (positive or negative) and taking the limit when δt tends to 0 :

$\frac{d}{dt}(u \circ \eta_1)(\tau) = \eta_2(\tau)\dot\eta_1(\tau)$ and by integration :

$$\int_a^b \eta_2(\tau)\dot\eta_1(\tau)d\tau = \int_a^b \frac{d}{dt}(u \circ \eta_1)(\tau)d\tau = u(\eta_1(b)) - u(\eta_1(a)) = 0.$$

3 Existence of subsolutions

Let us recall some well-known facts concerning the weak K.A.M. solutions, that the reader could find in [7] :

1. if $H : T^*M \to \mathbb{R}$ is a Tonelli Hamiltonian defined on the cotangent bundle of a compact and connected manifold M, there exists a unique $c \in \mathbb{R}$ such that the equation $H(q, du(q)) = c$ has weak K.A.M. solutions;

2. a weak K.A.M. solution of this previous equation is a semi-concave function $u\ : M \to \mathbb{R}$ such that, at every point of q of M where u has a derivative : $H(q, du(q)) = c$;
3. if $u\ : M \to \mathbb{R}$ is a weak K.A.M. solution, at every $q \in M$, for every super-differential p of u at q, we have : $H(q,p) \leq c$, i.e : $\forall (q,p) \in \mathcal{E}(u), H(q,p) \leq c$;
4. a subsolution for H is a semi-concave function $u\ : M \to \mathbb{R}$ such that at every point where u is derivable, we have : $H(q, du(q)) \leq c$ where c is the same as in item 1.

Let us now prove corollary 2. The function $H\ : T^*M \to \mathbb{R}$ being a Tonelli Hamiltonian, there exists at least one weak K.A.M. solution $u\ : M \to \mathbb{R}$. Then, by theorem 1, there exists $\varepsilon > 0$ such that for every $t \in]0, \varepsilon]$, $\varphi_{-t}(\mathcal{E}(u))$ is an exact Lagrangian Lipschitz graph. Hence there exists a $C^{1,1}$ function $v_t\ : M \to \mathbb{R}$ such that $\varphi_{-t}(\mathcal{E}(u))$ is the graph of dv_t. Then, for every $q \in M$, we have : $H(q, dv_t(q)) = H(\varphi_t(q, dv_t(q))) \leq c$ because $\varphi_t(q, dv_t(q)) \in \mathcal{E}(u)$ and u is a weak K.A.M. solution. We have proved that v_t is a $C^{1,1}$ subsolution.

4 Proof of proposition 3

We denote by $L\ : TM \to \mathbb{R}$ the Lagrangian associated to the Tonelli Hamiltonian H. We recall the definition of the semi-group $(\breve{T}_t)_{t>0}$:

$$\breve{T}_t u(q) = \max_{q' \in M} (u(q') - A_t(q, q')) \quad \text{where} \ : \quad A_t(q, q') = \min_{\gamma} \int_0^t L(\gamma(s), \dot{\gamma}(s)) ds$$

where the minimum is taken on the set of curves $\gamma \in C^2([0,t], M)$ which satisfy $\gamma(0) = q$ and $\gamma(t) = q'$.

P. Bernard proved in [2] that if u is semi-concave, for $t > 0$ small enough, $\breve{T}_t u$ is $C^{1,1}$. We choose such a t. Let us recall that there exists a constant $K_t = K > 0$ (see [1]) such that the function A_t is K-semi-concave (in the two variables). Moreover, for every $\gamma\ : [0,t] \to M$ joining q to q' which minimizes the Lagrangian action, $(-\frac{\partial L}{\partial v}(q, \dot{\gamma}(0)), \frac{\partial L}{\partial v}(q', \dot{\gamma}(t)))$ is a K-super-differential of A_t at (q, q').

Let us consider $q_0 \in M$ and let us use the notation : $v = \breve{T}_t u$. The function v being $C^{1,1}$, q_0 is a point of differentiability of v. There exists $q_1 \in M$ such that : $v(q_0) = u(q_1) - A_t(q_0, q_1)$. Then :

$$\forall q \in M, v(q) \geq u(q_1) - A_t(q, q_1) = v(q_0) + A_t(q_0, q_1) - A_t(q, q_1)$$

hence (in chart) :

$$\forall q \in M, v(q) - v(q_0) \geq A_t(q_0, q_1) - A_t(q, q_1) \geq \frac{\partial L}{\partial v}(q_0, \dot{\gamma}(0))(q_0 - q) - K\|q_0 - q\|^2$$

where $\gamma\ : [0,t] \to M$ is any C^2-arc minimizing joining q_0 to q_1.
As v is differentiable at q_0, it has at most one K-sub-differential and then the minimizing arc γ is unique. It implies (see the second appendix of [1]) that A_t is differentiable at (q_0, q_1). And we have :

$$\forall q \in M, v(q_0) = u(q_1) - A_t(q_0, q_1) \geq u(q) - A_t(q_0, q).$$

Hence (in chart) :

$$\forall q \in M, \frac{\partial L}{\partial v}(q_1, \dot\gamma(t))(q - q_1) + K\|q_1 - q\|^2 \geq A_t(q_0, q) - A_t(q_0, q_1) \geq u(q) - u(q_1)$$

and then $\frac{\partial L}{\partial v}(q_1, \dot\gamma(t))$ is a super-differential of u at q_1.
Finally, we have proved that :

- $dv(q_0) = \frac{\partial L}{\partial v}(q_0, \dot\gamma(0))$;
- $\frac{\partial L}{\partial v}(q_1, \dot\gamma(t))$ is a K-super-differential of u;

therefore : $\mathcal{G}(v) \subset \varphi_{-t}(\mathcal{E}(u))$. We have proved before that for $t > 0$ small enough, $\varphi_{-t}(\mathcal{E}(u))$ is a graph; hence necessarily we have $\mathcal{G}(v) = \varphi_{-t}(\mathcal{E}(u))$, i.e. $\mathcal{G}(\breve{T}_t u) = \varphi_{-t}(\mathcal{E}(u))$.
Remark. Let us note that the section 2.1 and the fact that $\mathcal{G}(\breve{T}_t u) \subset \varphi_{-t}(\mathcal{E}(u))$ implies directly that $\pi \circ \varphi_t(\mathcal{E}(u)) = M$ and give another proof if the subsection 2.2.

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